OBSTRUCTIONS FOR CONSTRUCTING G-EQUIVARIANT FIBRATIONS

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By
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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

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ABSTRACT

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August, 2011

Let G be a finite group and $\mathcal{H}$ be a family of subgroups of G which is closed under conjugation and taking subgroups. Let B be a G-CW-complex whose isotropy subgroups are in $\mathcal{H}$ and let $\mathcal{F} = \{F_H\}_{H \in \mathcal{H}}$ be a compatible family of H-spaces. A G-fibration over B with fiber $\mathcal{F} = \{F_H\}_{H \in \mathcal{H}}$ is a G-equivariant fibration $p : E \to B$ where $p^{-1}(b)$ is G_b-homotopy equivalent to F_{G_b} for each $b \in B$. In this thesis, we develop an obstruction theory for constructing G-fibrations with fiber $\mathcal{F}$ over a given G-CW-complex B. Constructing G-fibrations with a prescribed fiber $\mathcal{F}$ is an important step in the construction of free G-actions on finite CW-complexes which are homotopy equivalent to a product of spheres.

In this thesis we also consider the following question: For which finite groups the Euler class of the spherical fibration of the reduced regular representation is non-zero? This question was raised by Reiner and Webb in [18] and we answer this question completely.

Keywords: Bredon cohomology, equivariant fibration, equivariant quasi-fibration, Euler class, obstruction theory, orbit category.

ÖZET

G-EKUVARYANT LİFLEMELER ÜRETMENİN ENGEL TEORİSİ

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G sonlu bir grup, $\mathcal{H}$ ise elemanları G'nin altgruplarından oluşan, konjugasyon ve altgrup alma işlevleri altında kapalı bir aile olsun. B, izotropi altgrupları $\mathcal{H}$'in elemanları olan bir G-CW-kompleks ve $\mathcal{F} = \{F_H\}_{H\in\mathcal{H}}$, H-uzaylarından oluşan uyumlu bir aile olsun. B üzerinde tanımlı ve lifi $\mathcal{F} = \{F_H\}_{H\in\mathcal{H}}$ olan G-liflemesi, B'nin herhangi bir b elemanının ters görüntüsünün F_{G_b}'ye homotopi eşdeğer olduğu G-ekuvaryant bir liflemedir. Bu tezde, verilmiş bir G-CW-kompleksi üzerine, lifi $\mathcal{F} = \{F_H\}_{H\in\mathcal{H}}$ olan bir G-liflemesi oluşturmanın engel teorisini geliştirdik. Lifi $\mathcal{F}$ olan G-liflemeleri oluşturmak, üzerinde serbest G-etkisi olan ve kürelerin çarpımına homotopi eşdeğer olan sonlu CW-komplekslerin üretiminde önemli bir basamaktır.

Bu tezde, indirgenmiş düzenli temsillerin küresel liflemesinin Euler sınıfının hangi sonlu gruplar için sıfır olmadığı sorusunu da ele aldık. Bu soru Reiner ve Webb [18] tarafından sorulmuştu ve biz bu soruyu tam olarak cevapladık.

Anahtar sözcükler: Bredon kohomoloji, ekuvaryant lifleme, ekuvaryant yarılifleme, Euler sınıfı, engel teorisi, yörünge kategorisi.

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Contents

Chapter 1

Introduction

In 1925, Hopf stated a problem which is later called the topological spherical space form problem: Classify all closed manifolds with universal cover $\mathbb{S}^n$ for $n > 1$. This is equivalent to finding all finite groups which can act freely on a sphere, since closed manifolds with universal cover $\mathbb{S}^n$ are precisely the quotients of $\mathbb{S}^n$ by a free action of a finite group. One variant of this problem is solved by Swan [21]. He proved that a finite group acts freely on a finite complex homotopy equivalent to a sphere if and only if it has periodic cohomology. By using Swan's construction, the topological spherical space form problem has been solved completely by Madsen-Thomas-Wall [17]. It turns out that a finite group G acts freely on a sphere if and only if G has periodic cohomology and any element of order 2 in G is central.

One of the generalizations of this problem is to classify all finite groups which can act freely on a finite CW-complex homotopy equivalent to a product of k-spheres $\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_k}$ for some $n_1, \ldots, n_k$. Recently, Adem and Smith [1] gave a homotopy-theoretic characterization of cohomological periodicity and as a corollary they obtained a tool to construct free group actions on CW-complexes homotopy equivalent to a product of spheres. More precisely, they have shown that a connected CW-complex X has periodic cohomology if and only if there is a spherical fibration over X with a total space E that has a homotopy type of a finite dimensional CW-complex. As a consequence they proved that if X is a finite dimensional G-CW-complex whose all isotropy subgroups have periodic cohomology then there is a finite dimensional CW-complex Y with a free G-action such that $Y \simeq \mathbb{S}^n \times X$. As remarked in [1], the second result can also be obtained using the techniques given by Connolly and Parassidis in [6]. More recently, Klaus [13] proved that every p-group of rank 3 acts freely on a finite CW-complex homotopy equivalent to a product of three spheres by using similar techniques.

The method used by Connolly and Prassidis [6] is to construct a spherical fibration inductively over the skeleta by dealing with cells in each dimension separately. This is a standard strategy in obstruction theory. Note that if there is a G-spherical fibration over the n-th skeleton of the CW-complex, then its restriction to the boundary of each $(n+1)$-cell σ will be a G_σ-fibration with fiber F where G_σ is the isotropy subgroup of σ. Associated to this G_σ-fibration over $\partial\sigma$, there is a classifying map from $\partial\sigma$ to the space $B\mathrm{Aut}_{\mathrm{G}_\sigma}F$ where $\mathrm{Aut}_{G_\sigma}F$ is the topological monoid of self G_σ-homotopy equivalences of F. Combining the attaching map of σ with the classifying map gives us an element in the n-th homotopy group of $B\mathrm{Aut}_{\mathrm{G}_\sigma}F$. Therefore we obtain a cellular cochain which assigns a homotopy class in $\pi_n(B\mathrm{Aut}_{\mathrm{G}_\sigma}F)$ to each $(n+1)$-cell. This cochain vanishes if and only if the G-fibration over n-skeleton extends to a G-fibration over $(n+1)$-skeleton. In some cases, this cochain can be killed by taking fiber joins. Using this idea, Ünlü [27] gives a concrete cell-by-cell construction of G-spherical fibrations in his thesis.

In obstruction theory, one often has obstructions as cohomology classes which indicates when a construction can be performed on the next skeleton after some modifications. In other words, the cohomological obstruction class vanishes if and only if the restriction of the construction to the $(n-1)$-skeleton extends over the $(n+1)$-skeleton. Having a cohomological obstruction is better than having a cochain class as an obstruction since a cohomology class is more likely to be zero. In this thesis, we find cohomological obstructions for constructing G-fibrations and prove the following theorem.

Theorem 1.0.1 *Let G be a finite group and $\mathcal{H}$ be a family of subgroups of G which is closed under conjugation and taking subgroups. Let B be a G-CW-complex whose isotropy subgroups are in $\mathcal{H}$ and let $\mathcal{F} = \{F_H\}_{H \in \mathcal{H}}$ be a compatible family of H-spaces. Let $n \geq 1$ and $p : E_n \to B^n$ be a G-fibration over the n-th skeleton of B with fiber $\mathcal{F} = \{F_H\}_{H \in \mathcal{H}}$ where F_H is a finite H-CW-complex.*

1. *There is a cocycle $\alpha_p \in C^{n+1}(B; \underline{\pi}_n)$ which vanishes if and only if p extends to a G-fibration over B^{n+1} with a total space G-homotopy equivalent to a G-CW-complex.*

2. *The cohomology class $[\alpha_p] \in H^{n+1}(B; \underline{\pi}_n)$ vanishes if and only if the G-fibration $p|_{B^{n-1}} : p^{-1}(B^{n-1}) \to B^{n-1}$ extends to a G-fibration over B^{n+1} with a total space G-homotopy equivalent to a G-CW-complex.*

Moreover if B is a finite G-CW-complex then the total space of the obtained fibration has the homotopy type of a finite G-CW-complex whenever E_n has the homotopy type of a finite G-CW-complex.

To prove this theorem we first define an obstruction cochain in chain complex of Bredon cohomology and show that it is a cocyle, which we call the obstruction cocycle. Then we show that the difference of obstruction coycles of any two extensions of the G-fibration $p|_{B^{n-1}}$ is the coboundary of a cochain called difference cochain. If there is an extension of $p|_{B^{n-1}}$ to a G-fibration over B^{n+1}, then the obstruction cocycle of the restriction of this extension to B^n vanishes and hence the obstruction cocyle of p is a coboundary. Therefore it represents a cohomology class which vanishes. This proves the "if" direction of the above theorem.

For the "only if" direction it suffices to show that every cochain d there is a G-fibration q over B^n with $q|_{B^{n-1}} = p|_{B^{n-1}}$ such that d is the difference cochain of the extensions p and q of $p|_{B^{n-1}}$. Here the most technical part is the construction of a G-fibration q with these properties. That is because it is not clear how to glue G-fibration $p|_{B^{n-1}}$ with G-fibrations over the n-cells corresponding to the cochain d. For quasifibrations it suffices to take the adjunction of the total spaces to glue two quasibrations over different base. However, in order to obtain a fibration one needs to put some tubes between total spaces of these G-fibrations to create enough space to deal with G-homotopies (See Section 2.2). In Section 2.5, we give a method to glue G-fibrations over different base spaces by generalizing a construction due to Tulley [24].

Two fibrations p_1 and p_2 over B are said to be strongly fiber homotopy equivalent if there is a fibration $p : E \to B \times I$ such that $p|_{B\times\{0\}} = p_1$ and $p|_{B\times\{1\}} = p_2$. This notion was first introduced by Tuley in [24]. She also showed that if two fibrations are fiber homotopy equivalent then they are strong fiber homotopy equivalent. The equivariant analogue of the fibration p will produce the tube we need to construct the G-fibration q with desired properties.

Theorem 1.0.1 is particularly useful for constructing G-fibrations over base spaces with finite dimensional Bredon cohomology. The classifying space $E_{\mathcal{P}}(G)$ of G relative to the family $\mathcal{P}$ of p-subgroups is one example of such spaces. Its Bredon cohomology groups vanishes after degree $r = \mathrm{rk}_P(G)$ for any p-local coefficient system. Moreover applying Borel construction to a given G-equivariant fibration over $E_{\mathcal{P}}(G)$ yields a fibration over BG. One can use these properties to find a detection family for deciding when a cohomology class is an Euler class of a spherical fibration over BG.

In the last chapter of the thesis, we do some calculations involving Euler class of spherical fibrations in order to answer the problem posed by Reiner and Webb. The original problem is stated as follows: The subset complex $\Delta(G)$ of a finite group G is defined as the simplicial complex whose simplices are nonempty subsets of G. The oriented chain complex of $\Delta(G)$ gives a $\mathbb{Z}G$-module extension of $\mathbb{Z}$ by $\widetilde{\mathbb{Z}}$ where $\widetilde{\mathbb{Z}}$ is a copy of integers on which G acts via the sign representation of the regular representation. The extension class $\zeta_G \in \mathrm{Ext}^{|G|-1}_{\mathbb{Z}G}(\mathbb{Z}, \widetilde{\mathbb{Z}})$ of this extension is called the Ext class or the Euler class of the subset complex $\Delta(G)$. This class was first introduced by Reiner and Webb [18] who also raised the following question: For which finite groups G the Euler class ζ_G is nonzero?

In my master thesis, we considered the mod 2 reduction $\bar{\zeta}_G$ of the Euler class of the subset complex and we showed that when G is a 2-group, $\bar{\zeta}_G$ is non-zero if and only if $G \cong (\mathbb{Z}/2)^n \times \mathbb{Z}/4$. In this thesis we answer Reiner and Webb's question completely and obtain the following theorem.

Theorem 1.0.2 *Let G be a finite. Then, ζ_G is nonzero if and only if G is either an elementary abelian p-group or is isomorphic to $\mathbb{Z}/9$, $\mathbb{Z}/4\times\mathbb{Z}/4$, or $(\mathbb{Z}/2)^n\times\mathbb{Z}/4$ for some integer $n \geq 0$.*

This theorem is proved in the thesis in two parts as Theorem 4.0.1 and Theorem 4.0.2. To prove this theorem, we first show that ζ_G is zero when G is a nonabelian group and then we calculate ζ_G for specific abelian groups. The key ingredient in the proof is an observation by Mandell which says that the Euler class of the subset complex $\Delta(G)$ is equal to the (twisted) Euler class of the augmentation module of the regular representation of G.

This thesis is organized as follows:

In Chapter 2, we introduce necessary background material on G-equivariant fibrations and G-equivariant quasifibrations. Then we construct the universal G-fibration which classifies G-fibrations over a CW-complex with trivial G-action, up to G-fiber homotopy equivalence. We conclude this chapter by providing a way to glue fibrations over different base. This is done by generalizing a construction due to Tulley [24].

In Chapter 3, we develop an obstruction theory for constructing G-fibrations over G-CW-complexes and we prove the main theorem of the thesis. Since our obstructions lie in Bredon cohomology, we begin this chapter with the discussion of Bredon cohomology and the classical equivariant obstruction theory.

In the last chapter, we calculate the Euler classes of spherical fibrations associated to the reduced module of the regular representation. Using these calculations we completely solve a problem posed by Reiner and Webb in [18]. The results in this chapter were published in [11].

Chapter 2

Equivariant Fibration Theory

Given a finite group G, two main results of this chapter are to construct a universal classifying space for G-fibrations over a base with trivial G-action and to construct new G-fibrations out of given ones by glueing them over different base. These constructions will be used when we are developing an obstruction theory for constructing G-fibrations.

Let B be a CW-complex with trivial G-action. Let $\mathrm{Aut}_G F_G$ denote the monoid of the G-equivariant self homotopy equivalences of the finite G-CW complex F_G. In Section 2.3, we show that homotopy classes of maps from B to $B\mathrm{Aut}_G F_G$ classify G-equivariant fibrations over B up to G-fiber homotopy equivalence. This is the generalization of the analogous result obtained by Stasheff [20] for non-equivariant fibrations. We use the same techniques and ideas from [20]. In both cases the classifying space constructions yield universal quasifibrations which are needed to be replaced by fibrations. For this reason we devote first two sections of this chapter to preliminaries on G-equivariant fibrations and G-equivariant quasifibrations.

Two fibrations $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ are said to be *strongly fiber homotopy equivalent* if there is a fibration $q : E \to X \times I$ such that $q|_{B\times\{0\}} = p_1$ and $q|_{B\times\{1\}} = p_2$. This notion was first introduced by Tulley [24] for non-equivariant fibrations to extend a given fibration to a larger base space. For example, she proved that a fiber homotopically trivial fibration over B can be extended to the cone $C(B)$ of B. She obtained such a statement as a corollary to the fact that the property of being fiber homotopy equivalent coincides with the property of being strong fiber homotopy equivalent. In [14], Langston also observed this result for fibrations over a metric space whose total spaces are separable metric ANR's.

In the last two sections of this chapter, we extend the notion of strong fiber homotopy equivalence to G-equivariant fibrations and by using similar arguments we show the equivalence of the properties of being strongly G-fiber homotopy equivalent and G-fiber homotopy equivalent. As a consequence, we obtain a way to glue G-fibrations over different base, which have G-fiber homotopic restrictions.

2.1 G-fibrations

Definition 2.1.1 *A G-map $p : E \to B$ is called a G-fibration if it has G-homotopy lifting property for every G-space X, that is, given G-maps $h : X \to E$ and $H : X \times I \to B$ such that $H|_{X\times\{0\}} = p \circ h$, there exists a G-map $\widetilde{H} : X \times I \to E$ which makes the following diagram commute:*

$$\begin{tikzcd} X \times \{0\} \arrow[r, "h"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & B \end{tikzcd}$$

Proposition 2.1.1 *Let $p : E \to B$ be a G-fibration. Then*

1. *$p^{-1}(B') \to B'$ is a G-fibration for every G-subspace $B' \subseteq B$,*
2. *$p : E \to B$ is an H-fibration for every $H \leq G$,*

3. *$p^H : E^H \to B^H$ is a fibration for every $H \leq G$,*

4. *$p^{-1}(B^H) \to B^H$ is an $N_G(H)$-fibration for every $H \leq G$.*

Proof: See [16].

Proposition 2.1.2 *Let $p : E \to B$ be a G-fibration and let $D \subseteq E$. If $r : E \to D$ is a G-retract such that $p|_D \circ r(e) = p(e)$ for every $e \in E$ then $p|_D : D \to B$ is a G-fibration.*

Proof: Let $H : X \times I \to B$ and $h : X \to D$ be given G-maps with $H|_{X\times\{0\}} = p \circ h$. Since p is a G-fibration, there is a G-map $\bar{H} : X \times I \to E$ which makes the following diagram commute:

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "h"] \arrow[d, hook] & D \arrow[r, hook, "i"] \arrow[d, "p|_D"] & E \arrow[l, bend right, "r"'] \arrow[d, "p"] \\
X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] \arrow[urr, "\bar{H}"'] & B \arrow[r, equal] & B
\end{tikzcd}$$

Then the G-map $\widetilde{H} : X \times I \to D$ given by $\widetilde{H}(x,t) = r\bar{H}(x,t)$ makes the first diagram commute. Indeed $p|_D(\widetilde{H}(x,t)) = p|_D(r\bar{H}(x,t)) = p\bar{H}(x,t) = H(x,t)$ and $\widetilde{H}(x,0) = r\bar{H}(x,0) = (r \circ i)h(x) = h(x)$. $\square$

For any G-map $p : E \to B$, there is a universal testing diagram

$$\begin{tikzcd}
\Omega_p \times \{0\} \arrow[r, "\pi_1"] \arrow[d, hook] & E \arrow[d, "p"] \\
\Omega_p \times I \arrow[r, "H_{\Omega_p}"'] & B
\end{tikzcd}$$

for being a G-fibration where $\Omega_p = \{(e,\omega) \in E \times B^I | \ p(e) = \omega(0)\}$, π_1 is the projection to the first coordinate and, $H_{\Omega_p} : \Omega_p \times I \to B$ is defined by $H_{\Omega_p}((e,\omega),t) = \omega(t)$. Once p has G-HLP for this diagram, it has G-HLP for arbitrary G-space X. Indeed, we can consider any pair of G-maps $H : X \times I \to B$ and $h : X \to E$ with $H|_{X\times\{0\}} = h$ as a point $(h(x), \omega_H) \in \Omega_p$ where $\omega_H(t) = H(x,t)$ in Ω_p. Then $\widetilde{H} : X \times I \to E$ given by $\widetilde{H}(x,t) = H_{\Omega_p}(h(x),\omega_H)$ satisfies the relations $p \circ \widetilde{H} = H$ and $\widetilde{H}_0 = h$. Note that $\widetilde{H}_\Omega$ with these properties exists if and only if there is a G-map $\lambda : \Omega_p \to E^I$ such that

$$\lambda(e,\omega)(0) = e \text{ and } p(\lambda(e,\omega)(t)) = \omega(t).$$

We call such a G-map *G-lifting function* for $p : E \to B$ following the non-equivariant fibration theory.

Proposition 2.1.3 *A G-map $p : E \to B$ is a G-fibration if and only if there is a G-lifting function $\lambda : \Omega_p \to E^I$.*

A G-lifting function is called *regular* if it lifts constant paths to constant paths. In [12], Hurewicz proves that there is a regular lifting function for a fibration over B when B is a metric space.

Lemma 2.1.1 *If B is a metric space then every G-fibration over B admits a regular G-lifting function.*

Proof: Without loss of generality, we can assume that the metric on B is G-invariant. Indeed for every metric $d : B \times B \to \mathbb{R}$, there is an associated G-invariant metric d_G which is obtained by taking average of G-actions this means that d_G is defined by $d_G(x,y) = \frac{1}{|G|}\sum_{g\in G} d(gx,gy)$ for any $x, y \in B$.

For a G-invariant metric, let $d' : B^I \to I$ be defined by $d'(\omega) = \max\{\text{diam}(\omega(I)), 1\}$. Let λ be a G-lifting function for p. Then the map $\lambda' : \Omega_p \to E^I$ given by $\lambda'(e,\omega)(t) = \lambda(e,\omega')(d'(\omega)t)$ where

$$\omega'(s) = \begin{cases} \omega(\frac{s}{d'(\omega)}), & 0 \le s < d'(\omega); \\ \omega(1), & d'(\omega) \le s \le 1 \end{cases}$$

is a regular G-lifting function. □

A G-map $f : p_1 \to p_2$ between G-fibrations over the base B is called a *G-fiber map* if it satisfies the relation $p_2 \circ f = p_1$. A *G-fiber homotopy* between G-fiber maps $f : p_1 \to p_2$ and $g : p_1 \to p_2$ is a G-homotopy $H : E_1 \times I \to E_2$ with $p_2(H(e,t)) = p_1(e)$ for every $e \in E_1$ and $t \in I$. Now we can define G-fiber homotopy equivalence as usual.

Definition 2.1.2 *Let $p_i : E_i \to B$ be a G-fibration for $i = 1, 2$. We say p_1 and p_2 are G-fiber homotopy equivalent if there exist G-fiber maps $f : p_1 \to p_2$ and $g : p_2 \to p_1$ such that $f \circ g$ is G-fiber homotopic to id_{E_2} and $g \circ f$ is G-fiber homotopic to id_{E_1} via G-fiber homotopies. We write $p_1 \simeq p_2$. The G-maps f and g are called G-fiber homotopy equivalences.*

For a non-equivariant fibrations, Dold [8] showed that a fiber preserving map which is also a homotopy equivalence is a fiber homotopy equivalence. Although the same proof applies to equivariant fibrations, we provide it here for completeness.

Theorem 2.1.1 *Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be G-fibrations. Then a G-fiber map $f : E_1 \to E_2$ is a G-fiber homotopy equivalence if and only if it is a G-homotopy equivalence.*

Proof: Let $g : E_2 \to E_1$ be the G-homotopy inverse and let $h : E_2 \times I \to E_2$ be the G-homotopy equivalence between $f \circ g$ and id_{E_2}. Since $p_2 \circ h : E_2 \times I \to B$ satisfies the relation $p_1 \circ g = p_2 h|_{E_2 \times \{0\}}$, we can lift it to a G-map $H : E_2 \times I \to E_1$. Then $g' = H(-, 1) : E_2 \to E_1$ is a fiber preserving G-map with $fg' \simeq \mathrm{id}_{E_2}$ and $g'f \simeq \mathrm{id}_{E_1}$. To conclude that f is a G-fiber homotopy equivalence we also need to replace G-homotopies with fiber preserving ones.

Note that fg' is homotopic to id_{E_2} via the G-homotopy $\bar{h} : E_2 \times I \to E_2$ where

$$\bar{h}(e,t) = \begin{cases} fH(e, 1-2t), & 0 \le t \le \frac{1}{2}; \\ h(e, 2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

Define $F : E_2 \times I \times I \to B$ by

$$F(e,s,t) = \begin{cases} p_2 f H(e, 1-2s(1-t)), & 0 \le s \le \frac{1}{2}; \\ p_2 h(e, 1-2(1-s)(1-t)), & \frac{1}{2} \le s \le 1. \end{cases}$$

Since $F|_{E_2 \times I \times \{0\}} = p_2 \bar{h}$, there is a G-map $\widetilde{F} : E_2 \times I \times I \to B$ with $p_2(\widetilde{F}) = F$ and $\widetilde{F}|_{E_2 \times I \times \{0\}} = \bar{h}$. Therefore the G-map $\bar{F} : E_2 \times I \to E_2$ defined by

$$\bar{F}(e,t) = \begin{cases} \widetilde{F}(e,0,3t), & 0 \le t \le \frac{1}{3}; \\ \widetilde{F}(e,3t-1,1), & \frac{1}{3} \le t \le \frac{2}{3}; \\ \widetilde{F}(e,1,3(1-t)), & \frac{2}{3} \le t \le 1. \end{cases}$$

is the desired G-fiber homotopy between fg' and id_{E_2}. Similarly, one can show that $g'f$ and id_{E_1} are G-fiber homotopic. $\Box$.

One can associates to every G-map $p : E \to B$ a G-fibration

$$\mathrm{Hur}_G(p) : \mathrm{Hur}_G(E) \to B$$

by letting $\mathrm{Hur}_G(E) = \Omega_p$ and $\mathrm{Hur}_G(p)(e,\omega) = \omega(1)$. When $p : E \to B$ is itself a G-fibration, p is G-fiber homotopy equivalent to $\mathrm{Hur}_G(p)$. To see this, let $\lambda : \Omega_p \to E^I$ be a G-lifting function for p. Define $f : \mathrm{Hur}_G(E) \to E$ and $g : E \to \mathrm{Hur}_G(E)$ by $f(e,\omega) = \lambda(e,\omega)(1)$ and $g(e) = (e, *_{p(e)})$. Then $H_1 : \mathrm{Hur}_G(E) \times I \to \mathrm{Hur}_G(E)$ defined by $H_1(e,\omega)(t) = (\lambda(e,\omega)(t), \omega^t)$ where

$$\omega^t(s) = \begin{cases} \omega(s+t), & 0 \le s+t \le 1; \\ \omega(1), & \text{otherwise.} \end{cases} \tag{2.1}$$

and $H_2 : E \times I \to E$ defined by $H_2(e,t) = \lambda(e,\omega)(t)$ are the required fiber preserving G-maps such that $g \circ f \simeq_{H_1} \mathrm{id}|_{\mathrm{Hur}_G(E)}$ and $f \circ g \simeq_{H_2} \mathrm{id}_E$. This is a very standard result in non-equivariant fibration theory, see [32].

Moreover, the total space E is a strong G-deformation retract of $\mathrm{Hur}_G(E)$ via the map $H : \mathrm{Hur}_G(E) \times I \to \mathrm{Hur}_G(E)$ defined by $H((e,\omega),t) = (e,\omega_t)$ where $\omega_t(s) = \omega((1-t)s)$. However it is not fiber preserving.

Proposition 2.1.4 *If $p : E \to B$ is a G-fibration over a metric space B, then there is a fiber preserving G-strong deformation retraction of* $\mathrm{Hur}_G(E)$ *onto E.*

Proof: Let $r : \mathrm{Hur}_G(E) \to E$ be the retraction defined by $r(e,\omega) = (\lambda(e,\omega)(1), *_{\omega(1)})$ where λ is a regular G-lifting function for $\mathrm{Hur}_G(p)$. Note that r is a fiber preserving retraction of E. Define $H : \mathrm{Hur}_G(E) \times I \to \mathrm{Hur}_G(E)$ by $H(e,\omega)(t) = (\lambda(e,\omega)(t), \omega^{1-t})$ where ω^t is given as above. Since λ lifts constant paths to constant paths, H is a strong G-deformation retraction with $H(e,0) = r$. As in the proof of Theorem 2.1.1, we can replace it with a G-fiber homotopy $\widetilde{H}$. If we define all the liftings in question by using λ then we obtain a G-map with the property $\widetilde{H}(e, *_{p(e)}, t) = (e, *_{p(e)})$. $\square$

Another way of constructing G-fibrations is taking pullbacks. Let $p : E \to B$ be a G-fibration and $f : X \to B$ be an arbitrary G-map. Then the induced map $f^*(p) : f^*(E) \to X$ is a G-fibration where $f^*(E) = \{(x,e) \in X \times E |\ f(x) = p(e)\}$ is the pullback of p and f. The following is a well-known result for G-fibrations, see [16].

Proposition 2.1.5 *Let $p : E \to B$ be a G-fibration. If f_0 and f_1 from X to B are G-homotopic then $f_0^*(p) \cong f_1^*(p)$.*

A covering $\mathcal{U}$ of G-invariant open sets of B is called *numerable G-covering* if $\mathcal{U}$ is locally finite, for every $U \in \mathcal{U}$, there is a G-map $f_U : B \to I$ such that $U = f_U^{-1}(0,1]$. We say a G-map $p : E \to B$ is a *numerable local G-fibration* if there is a numerable G-covering $\{U_i\}$ of B such that $p^{-1}U_i \to U_i$ is a G-fibration. We refer reader to Dold [8] for the proofs of the following theorems for non-equivariant fibrations.

Theorem 2.1.2 (Uniformization theorem) *Every numerable local G-fibration is a G-fibration.*

Theorem 2.1.3 *Let $p_i : E_i \to B$ be G-maps for $i = 1, 2$. Let G-map $f : E_1 \to E_2$ such that $p_2 f = p_1$. Suppose that $f_U : p_1^{-1}U \to p_2^{-1}U$ induced by f is a G-fiber homotopy equivalence for every U in some numerable G-covering $\mathcal{U}$. Then f is a G-fiber homotopy equivalence.*

2.2 G-quasifibrations

In this section, we give preliminary definitions and main properties of the equivariant G-quasifibration theory. Most of the results are stated without proofs. We refer reader to [29] and [30] for more details.

Roughly speaking a quasifibration is a fibration up to weak equivalence. More precisely, a map $p : E \to B$ is a quasifibration if $p_* : \pi_n(E, p^{-1}(b), e) \to \pi_n(B, b)$ is an isomorphism for every $b \in B, e \in p^{-1}(b)$ and for every n.

Definition 2.2.1 *A G-quasifibration is a G-map $p : E \to B$ between G-spaces such that $p^H : E^H \to B^H$ is a quasifibration for every subgroup H of G.*

Note that every G-fibration is a G-quasifibration. Unfortunately the converse is not true. A simple counterexample for this is the map $p : [0, 1] \to [0, \frac{2}{3}]$ which contracts the subinterval $[\frac{1}{3}, \frac{2}{3}]$ into a point $\{\frac{2}{3}\}$. Here p is not a fibration since we cannot lift the path $\omega : [0, \frac{2}{3}] \to [0, \frac{2}{3}]$ defined by $\omega(t) = \frac{2}{3} - t$ to a path $\widetilde{\omega} : I \to I$ starting at the point $\widetilde{\omega}(0) = 1$.

Definition 2.2.2 *An equivariant subspace $U \subset B$ is said to be G-distinguished if $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is a G-quasifibration.*

The following results about G-quasifibrations are easily follow from the non-equivariant cases by taking H-fixed points for every $H < G$.

Proposition 2.2.1 ([30], Lemma 2.3) *Let U be a G-distinguished subspace of B with respect to the G-map $p : E \to B$. If there are G-deformations h of B onto U and H of E onto $p^{-1}(U)$ such that $pH_1 = h_1 p$ and $H_1|_{p^{-1}\{b\}}$ is a G-weak equivalences for every $b \in B$, then p is a G-quasifibration.*

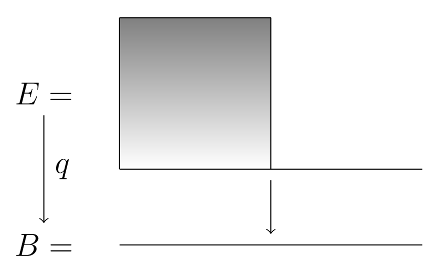
Let A, B be an equivariant subspaces of a G-space X. The triad $(X; A, B)$ is called G-excisive if X is the union of interiors of A and B.

Proposition 2.2.2 ([30], Lemma 2.6) *Let $(B; B_1, B_2)$ be an excisive G-triad such that B_1, B_2 and $B_1 \cap B_2$ are G-distinguished with respect to the G-map $p : E \to B$. Then p is a G-quasifibration.*

Corollary 2.2.1 *Let $p_i : E_i \to B_i$ be G-fibrations for $i = 1, 2$ and $B_1 \cap B_2 \neq \emptyset$. If $p_1|_{B_1 \cap B_2} \simeq p_2|_{B_1 \cap B_2}$, then there is a G-quasifibration over $B_1 \cup B_2$ which extends p_1 and p_2.*

Proof: Let $\phi : p_1^{-1}(B_1 \cap B_2) \to p_1^{-1}(B_1 \cap B_2)$ be the G-fiber homotopy equivalence. Then the G-map $q : E_1 \cup_\phi E_2 \to B_1 \cup B_2$ defined by $q|_{E_i} = p_i$ for $i \in \{1, 2\}$ is a G-quasifibration by the above proposition. $\square$

Unfortunately, the G-map q defined as in the proof of the above lemma is not always a G-fibration. For example, if $p_1 : I \times I \to I$ is the projection to the first coordinate and $p_2 : I \to I$ is the identity with trivial G-actions then $q : [0, 1] \times [0, 1] \cup [1, 2] \times \{0\} \to [0, 2]$



is not a G-fibration. That is because we can not lift the path $\omega : I \to [0, 2]$ defined by $\omega(t) = t + 1$ to a path $\widetilde{\omega} : I \to E$ with $\widetilde{\omega}(0) = (1, 1)$. Recall that if we apply Hur_G construction to q, we obtain a G-fibration $\mathrm{Hur}_G(p)$. However, $\mathrm{Hur}_G(p)$ extends p_1 and p_2 only up to fiber homotopy equivalence. In the following section we will construct a G-fibration which actually extends p_1 and p_2 under the assumption of the above corollary.

To construct the classifying space for G-fibrations over a CW-complex B with trivial G-action, we also need the following observation about G-quasifibrations. We refer reader to [8] for the proof of the non-equivariant version.

Proposition 2.2.3 *If B is the inductive limit of a sequence of G-distinguished subspaces $B_1 \subset B_2 \subset \cdots$ with respect to the G-map $p : E \to B$, then p is a G-quasifibration.*

2.3 Classifying spaces for G-fibrations

Let $[B, X]$ be the set of homotopy classes of maps from B into X and let $LF(B)$ be the set of fiber homotopy equivalence classes of fibrations over B with fibers of the homotopy of F. Let $\mathcal{C}$ be the category of CW-complexes and homotopy classes of maps and $\mathcal{S}$ be the category of sets and functions. In [20], Stasheff shows that when F is a finite CW-complex, the functors $[-, B\mathrm{Aut}F] : \mathcal{C} \to \mathcal{S}$ and $LF[-] : \mathcal{C} \to \mathcal{S}$ are naturally equivalent where $\mathrm{Aut}(F)$ is the topological monoid of self homotopy equivalences of F. In this section, we prove the following generalization of Stasheff's classification theorem by using the same techniques and ideas.

Theorem 2.3.1 *Let $\mathrm{Aut}_G(F_G)$ be the monoid of G-equivariant self homotopy equivalences of a finite G-CW-complex F_G. If B is a CW-complex with trivial G-action then there is a one-to-one correspondence between the set of G-fiber homotopy equivalence classes of G-fibrations $p : X \to B$ with fibers of the G-homotopy type of F_G and the set of homotopy classes of maps $B \to B\mathrm{Aut}_G(F_G)$.*

Throughout the section, let B be a CW-complex with trivial G-action. We first prove the following fact about G-fibrations over B which is used repeatedly in the proof of our main theorem. It is the generalization of the Proposition 0 in [20].

Proposition 2.3.1 (See [20], Proposition 0) *Let $p : E \to B$ be a G-fibration with fibers of the G-homotopy type of a G-CW-complex. Then the total space E has the G-homotopy type of a G-CW-complex.*

The proof of the proposition follows from the following easy observation.

Lemma 2.3.1 *Let $p : E \to B$ be a G-fibration with fibers of the G-homotopy type of a G-CW-complex F_G. If $E_{n-1} = p^{-1}(B^{n-1})$ has the G-homotopy type of a G-CW-complex so does $E_n = p^{-1}(B^n)$.*

Although the proof of the lemma is very similar to the proof of than that of the Proposition 1 in [20], we provide a proof here for the future references.

Proof: Without loss of generality, we can assume that the n-th skeleton B^n is obtained from B^{n-1} by attaching a single n-cell σ since the result is obtained from this case by attaching n-cells one by one. Let $f_\sigma : (\mathbb{D}^n, \mathbb{S}^{n-1}) \to (B^n, B^{n-1})$ be the classifying map of σ. Since $\mathbb{D}^n$ is contractible, $f_\sigma^*(p)$ is G-homotopy equivalent to the trivial G-fibration $F_G \times \mathbb{D}^n$. Let $\phi : F_G \times \mathbb{D}^n \to f_\sigma^*(E)$ be a G-fiber homotopy equivalence with G-fiber homotopy inverse $\psi : f_\sigma^*(E) \to F_G \times \mathbb{D}^n$. Let $\bar{\phi} = \phi|_{S^{n-1}}$.

Now consider $\mathbb{D}^n$ as a cone of $\mathbb{S}^{n-1}$. Define $s : F_G \times \mathbb{D}^n \to F_G \times \mathbb{D}^n$ by

$$s(y,(t,x)) = \begin{cases} (y,(x,0)), & 0 \le t \le \frac{1}{2}; \\ (y,(x,2t-1)), & \frac{1}{2} \le t \le 1. \end{cases}$$

Note that s is G-homotopy equivalent to the identity map. So $s \circ \psi$ is also a G-homotopy inverse of ϕ.

Let $f : E_{n-1} \cup_{\pi_2\bar{\phi}} F_G \times \mathbb{D}^n \to E_n$ be defined by $f|_{F_G \times \mathbb{D}^n} = \pi_2\phi$ and $f|_{E_{n-1}} = \mathrm{id}_{E_{n-1}}$. Clearly, f is well-defined. Let us define $g : E_n \to E_{n-1} \cup_{\pi_2\bar{\phi}} F_G \times \mathbb{D}^n$ extending the identity on E_{n-1} as follows. If $e \in E_n \backslash E_{n-1}$ then $p(e) \in B^n \backslash B^{n-1}$ and there is a unique $(x_e, t_e) \in \mathbb{D}^n$ such that $f_\sigma(x_e, t_e) = p(e)$. Let us define $g(e) = s \circ \psi((x_e, t_e), e)$. Since ψ preserves fibers, g is a well-defined continuous function. Clearly, f and g are mutually G-homotopy inverses relative to E_{n-1}. Therefore E_n is G-homotopy equivalent to $E_{n-1} \cup_{\pi_2\bar{\phi}} F_G \times \mathbb{D}^n$ relative to E_{n-1} and hence is of the G-homotopy type of a G-CW-complex. $\square$

Proof of Proposition 2.3.1: Following [20], let $E_\omega = \bigcup_{\sigma \in I} p^{-1}(\sigma)$ endowed with the weak topology where I is the set of closed cells of B. Let $i : E_\omega \to E$ be the identity map from weak topology to the original one. Let $p_\omega = p \circ i$. By Lemma 2.3.1, E_ω has the G-homotopy type of a G-CW-complex. Since E_ω is the G-deformation retract of $\mathrm{Hur}_G(E_\omega)$, $\mathrm{Hur}_G(E_\omega)$ also has the G-homotopy type of a G-CW-complex. Therefore it suffices to show that E has the same G-homotopy type with $\mathrm{Hur}_G(E_\omega)$ in order to prove the theorem.

Let $H : \mathrm{Hur}_G(E_\omega) \times I \to \mathrm{Hur}_G(E_\omega)$ be the G-deformation retraction and let $r = H_1$. Since $r \simeq \mathrm{id}|_{\mathrm{Hur}_G(E_\omega)}$, $p \circ i \circ r = p_\omega \circ r$ is G-homotopic to $\mathrm{Hur}_G(p_\omega)$. The G-homotopy $h : \mathrm{Hur}_G(E_\omega) \times I \to B$ between $p \circ i \circ r$ and $\mathrm{Hur}_G(p_\omega)$ yields a following commutative diagram

$$\begin{tikzcd}
\mathrm{Hur}_G(E_\omega) \times \{0\} \arrow[r, "i \circ r"] \arrow[d, hook] & E \arrow[d, "p"] \\
\mathrm{Hur}_G(E_\omega) \times I \arrow[r, "h"'] & B
\end{tikzcd}$$

Since $p : E \to B$ is a G-fibration, there is a G-lifting $\widetilde{h} : \mathrm{Hur}_G(E_\omega) \times I \to E$. Let $g = \widetilde{h}|_{\mathrm{Hur}_G(E_\omega)\times\{1\}}$. Since i is a weak G-homotopy equivalence, r is a G-homotopy equivalence and g is a G-fiber map, the restriction of g to the fibers is a weak G-homotopy equivalence.

Assume for a moment that the restriction of g to the fibers is a G-homotopy equivalence. Then by Theorem 2.1.3, g is a homotopy equivalence and hence E is G-homotopy equivalent to $\mathrm{Hur}_G(E_\omega)$ as desired. Since the fibers of $p : E \to B$ are all G-homotopy type of a G-CW-complex, to conclude that the restriction of g to the fibers is a G-homotopy equivalence, it suffices to show that the fibers of $\mathrm{Hur}_G(p_\omega)$ also has the G-homotopy type of a G-CW-complex.

For some $b \in B$, let $\varepsilon = \{\lambda : I \to \{b\} \times I \cup B \cup_p \mathrm{Hur}_G(E_\omega) \times I \mid \lambda(0) = (b, 1)$ and $\lambda(1) \in \mathrm{Hur}_G(E_\omega) \times \{1\}$. Since it is G-homotopic to $\mathrm{Hur}_G(E_\omega)$, ε has the G-homotopy type of a G-CW-complex. Let us show that $p^{-1}(b)$ is G-homotopic to ε. For this, let $f : p^{-1}(b) \to \varepsilon$ be defined for any $e \in E$ by

$$f(e)(t) = \begin{cases} (b, 1-2t), & 0 \le t \le \frac{1}{2}; \\ (e, 2t-1), & \text{otherwise.} \end{cases}$$

It is well-defined since $(e, 0) = p(e) = b = (b, 0)$. Let $\alpha : \varepsilon \to \mathrm{Hur}_G(E) \times \{1\}$ be defined by $\alpha(\lambda) = (\lambda(1), 1)$. Then $p \circ \alpha$ is G-homotopic to the projection π_b

to $b \in B$. Since p is a G-fibration there is a G-map $\widetilde{H} : \varepsilon \times I \to E$ making the following diagram commute:

$$\begin{tikzcd}
\varepsilon \times \{0\} \arrow[r, "\alpha"] \arrow[d, hook] & E \arrow[d, "p"] \\
\varepsilon \times I \arrow[r, "\phi"'] \arrow[ur, dashed, "\widetilde{\phi}"] & B
\end{tikzcd}$$

where ϕ is the G-homotopy between $p \circ \alpha$ and π_b. Then the G-map $g : \varepsilon \to p^{-1}(b)$ defined by

$$g(\lambda) = \widetilde{\phi}(\lambda, 1).$$

is the G-homotopy inverse of f. $\square$

Given a G-quasifibration $p : E \to B$ with fibers of the G-homotopy type of F_G, it is standard to construct the *associated G-principal map* as follows:

$$\begin{array}{c}
\mathrm{Prin}_G E = \{G\text{−maps } \varphi : F_G \to E \text{ which are } G\text{−homotopy} \\
\text{equivalence between } F_G \text{ and some fiber}\} \\
\Big\downarrow {\scriptstyle \mathrm{Prin}_G(p)} \\
B
\end{array}$$

where $\mathrm{Prin}_G(p)(\varphi) = p(\varphi(F_G))$. Here the G-action on the space $\mathrm{Prin}_G E$ is trivial.

Lemma 2.3.2 *Let $p : E \to B$ be a G-fibration over B with fiber F_G. Then $\mathrm{Prin}_G(p)$ is a fibration with fiber $\mathrm{Aut}_G(F_G)$.*

Proof: A commutative diagram of the form

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "f"] \arrow[d] & \mathrm{Prin}_G E \arrow[d] \\
X \times I \arrow[r, "H"] & B
\end{tikzcd}$$

yields a following commutative diagram of G-maps

$$\begin{tikzcd}
X \times F_G \times I \times \{0\} \arrow[r, "\bar{f}"] \arrow[d] & E \arrow[d] \\
X \times F_G \times I \times I \arrow[r, "\bar{H}"] & B
\end{tikzcd}$$

where $\bar{f}(x, y, t, 0) = f(x)(y)$ and $\bar{H}(x, y, t_1, t_2) = h(x, t_2)$. By G-HLP of p, there is a G-lifting $\widetilde{H} : X \times F_G \times I \times I \to E$ of $\bar{H}$. Then the map $\theta : X \times I \to \mathrm{Prin}_G E$ defined by $\theta(x, t)(y) = \widetilde{H}(x, y, 1, t)$ gives a homotopy lifting of H. Clearly, fibers of $\mathrm{Prin}_G(p)$ are G-homotopic to the space $\mathrm{Aut}_G(F)$. $\square$

The above lemma is the analogue of Lemma 7 of [20]. Stasheff also gave other ways of getting new fibrations or quasifibrations from the old ones. His aim was to obtain the universal fibration by applying these constructions repeatedly. We also use the analogous constructions to prove Theorem 2.3.1. One of these construction is the G-prolongation. Let $p : E \to B$ be a G-quasifibration with fibers of the G-homotopy type of F_G. We define the G-prolongation $\mathrm{Prol}_G(p)$ as

$$\begin{array}{c} \mathrm{Prol}_G E = (C\mathrm{Prin}_G E \times F) \cup_\nu E \\ \Big\downarrow \mathrm{Prol}_G(p) \\ \mathrm{Prol}_G B = C\mathrm{Prin}_G E \cup_{\mathrm{Prin}_G(p)} B \end{array}$$

where $\nu : \mathrm{Prin}_G E \times F \to E$ is defined by $\nu(\phi, y) = \phi(y)$ for any $(\phi, y) \in \mathrm{Prin}_G E \times F$.

Lemma 2.3.3 $\mathrm{Prol}_G E$ *is a G-quasifibration when p is.*

Proof: Let us consider the restrictions of $\mathrm{Prol}_G(p)$ to the following subspaces $A = \mathrm{Prin}_G E \times [1/3, 1]/_{(\phi,1)\simeq *}$ and $B = \mathrm{Prin}_G E \times [0, 2/3] \cup_{\mathrm{Prin}_G(p)} B$. Since

$$\mathrm{Prol}_G(p)|_{\mathrm{Prol}_G^{-1}(p)A} : \mathrm{Prin}_G E \times [1/3, 1] \times F_G/_{(\phi,1,y)\simeq(*,y)} \to A$$

and

$$\mathrm{Prol}_G(p)|_{\mathrm{Prol}_G^{-1}(p)(A\cap B)} : \mathrm{Prin}_G E \times [1/3, 2/3] \times F_G \to A \cap B = \mathrm{Prin}_G E \times [1/3, 2/3]$$

are G-projections and hence G-quasifibrations, A and $A \cap B$ are G-distinguished. By Proposition 2.2.2, it suffices to show that B is also a G-distinguished subset.

Note that E is a G-deformation retract of $\mathrm{Prol}_G^{-1}(p)(B)$ via the G-deformation retraction $H : \mathrm{Prol}_G^{-1}(p)(B) \times I \to \mathrm{Prol}_G^{-1}(p)(B)$ given by

$$H(\phi, s, y, t) = (\phi, s - st, y) \text{ and } H(e, t) = e.$$

Furthermore, the G-map $h : C \times I \to C$ with

$$h(\phi, s, t) = (\phi, s - st) \text{ and } h(b, t) = b$$

is a G-deformation retraction of B onto C. Since $H_1 : p^{-1}b \to p^{-1}(h_1(b)) = p^{-1}(b)$ is the identity map for every $b \in B$, and

$$\begin{aligned}
\mathrm{Prol}_G(p) \circ H(\phi, s, y, 1) &= \mathrm{Prol}_G(p)(\phi, 0, y) = \mathrm{Prol}_G(p)(\phi y) = p(\phi y) \\
&= \mathrm{Prin}_G(p)(\phi) = (\phi, 0) = h_1(\phi, s) \\
&= h_1 \circ \mathrm{Prol}(\phi, s, y),
\end{aligned}$$

C is G-distinguished by Proposition 2.2.1. $\square$

In light of the above lemma, we can inductively construct G-quasifibrations $q_n : E_n \to B_n$ from the given G-quasifibration $p : E \to B$ by taking $q_0 = p$ and q_n to be $\mathrm{Prol}_G(q_{n-1}) : \mathrm{Prol}_G(E_{n-1}) \to \mathrm{Prol}_G(B_{n-1})$. By Proposition 2.2.3, the limit of the G-quasifibrations q_n is a G-quasifibration. Following Stasheff we denote this quasifibration by $\mathrm{Ult_G(p)} : \mathrm{Ult}_G E \to \mathrm{Ult}_G B$.

Since $\mathrm{Prin}_G(C(\mathrm{Prin}_G E) \times F_G) \cong_G C(\mathrm{Prin}_G E) \times \mathrm{Aut}_G F_G$, we have the following analogue of Lemma 10 and Lemma 11 in [20].

Lemma 2.3.4 *If* $\mathrm{Prin}_G(p)$ *is a* G*-quasifibration so is the* G*-map*

$$\mathrm{Prin}_G(\mathrm{Prol}_G(p)) : \mathrm{Prin}_G(C(\mathrm{Prin}_G E) \times F_G \cup_\nu E) \to C(\mathrm{Prin}_G E) \cup_{\mathrm{Prin}_G(p)} B.$$

Moreover $\mathrm{Prin}_G(\mathrm{Ult}_G(p))$ *is a quasifibration with aspherical total space if* p *is a* G*-fibration.*

Let us now consider the G-fibration $\theta : F_G \to *$ where F_G is a finite G-CW-complex. Clearly the induced G-map $\mathrm{Hur}_G(\mathrm{Ult}_G(\theta))$ is a G-fibration with fibers of the weak homotopy type of the G-CW-complex F_G. On the other hand, B has the homotopy type of a CW-complex and hence $\mathrm{Prin}_G(F_G)$ has the homotopy type of CW-complex. Since F_G is compact, $\mathrm{Aut}_G F_G$ has the G-homotopy type of a G-CW-complex so does $\mathrm{Prol}_G(F_G)$ and $\mathrm{Prol}_G(B)$ by Proposition 2.3.1. Since

$\mathrm{Prin}_G(\mathrm{Prol}_G(F_G))$ is G-homeomorphic to $\mathrm{Prol}_G(\mathrm{Prin}_G(F_G))$, by repeating this argument we obtain that the spaces $\mathrm{Ult}_G(F_G)$ and $\mathrm{Hur}_G(B)$ have the G-homotopy type of a G-CW-complexes. By using the same arguments used in the proof 2.3, one can conclude that the fibers of $\mathrm{Ult}_G(\theta)$ has the G-homotopy type of the G-CW-complex F_G. For the sake of simplicity, we denote the G-fibration $\mathrm{Hur}_G(\mathrm{Ult}_G(\theta))$ by u_G. As in the standard theory, we denote the base space of this G-fibration by $B\mathrm{Aut}_G(F_G)$.

Every map $f : B \to B\mathrm{Aut}_G(F_G)$ induces a G-fibration $f^*(U_G)$ over B. Therefore in order to prove the classification theorem (Theorem 2.3.1), we need to construct the converse map which assigns every G-fibration over B to a homotopy class of a map from B to $B\mathrm{Aut}_G(F_G)$. For this, consider the following commutative diagram of G-fibrations:

$$\begin{tikzcd} F_G \arrow[r, "h"] \arrow[d, "\theta"'] & E \arrow[d, "p"] \\ b \arrow[r, hook, "j"'] & B \end{tikzcd}$$

for some $b \in B$. Applying Ult_G-construction yields a following commutative diagram of G-fibrations

$$\begin{tikzcd} \mathrm{Ult}_G F_G \arrow[r] \arrow[d, "u_G"'] & \mathrm{Ult}_G E \arrow[d, "\mathrm{Ult}_G(p)"'] \\ B\mathrm{Aut}_G F_G \arrow[r, "\mathrm{Ult}_G(j)"] & \mathrm{Ult}_G(B). \end{tikzcd}$$

Since $\mathrm{Prin}_G(\mathrm{Ult}_G E)$ and $\mathrm{Prin}_G(\mathrm{Ult}_G F_G)$ are aspherical, $\mathrm{Ult}_G(j)$ is a weak homotopy equivalence. Therefore the induced map

$$j_* : [B, B\mathrm{Aut}_G F_G] \to [B, \mathrm{Ult}_G B]$$

is an isomorphism. Note that different points form the same path-component induces the same isomorphism j_* up to homotopy. Therefore every G-fibration $p : E \to B$ gives a homotopy class $[f] \in [B, B\mathrm{Aut}_G(F_G)]$ where on each component $j_*[f]$ is the inclusion of B into $\mathrm{Ult}_G B$. We refer reader to [20] for more details.

In [2], Allaud proved a version of Stasheff's classification theorem by assuming F to have only the homotopy type of a CW-complex. To verify such a result he used the Brown representability theorem. For this reason, he worked in the category of based spaces and based maps and hence uses slightly different functor than that of Stasheff's.

More precisely, let $\mathcal{C}_0$ be the category whose objects are CW-complexes with base point and whose morphisms are homotopy classes of maps preserving base points and let $\mathcal{S}_0$ be the category of based spaces and based maps. According to [2], a fibration over $(B, b_o) \in \mathcal{C}$ is a pair (p, g) where $p : E \to B$ is a fibration over B and $g : F \to p^{-1}(b_o)$ is a fiber homotopy equivalence. Two fibration (p_1, g_1) and (p_2, g_2) are said to be equivalent if there is a fiber homotopy equivalence $f : p_1 \to p_2$ such that $f \circ g_1$ is homotopic to g_2.

Let $HF(B)$ be the set of equivalence classes of fibrations over (B, b_0) with base point (π_1, i) where $\pi_1 : B \times F \to B$ is the trivial fibration and $i : F \to F \times \{b_0\}$ is the inclusion. In [2], it is shown that $HF(-) : \mathcal{C}_0 \to \mathcal{S}_0$ is a homotopy functor. Therefore there is a space B_∞ such that the functors $HF()$ and $(-, B_\infty)$ are naturally equivalent via the equivalence obtained by taking induced fibrations from a universal one over B_∞. Similarly, we can extend this result to G-fibrations over a CW-complex with trivial G-action and with fibers of the homotopy type of a G-CW-complex F_G. This will let us to develop an obstruction theory for G-fibrations over a G-CW-complex with fibers of the homotopy type of a G-CW-complex as in Section 3.3.

2.4 Strong G-fiber homotopy equivalence

The property of being strong fiber homotopy equivalent is introduced in [24] and is studied in detailed in the PhD thesis of Langston [14]. In this section we generalize the notion of strong fiber homotopy equivalence to G-fibrations and we prove some basic results concerning it.

Definition 2.4.1 *Two G-fibration p_1 and p_2 are said to be strongly G-fiber homotopy equivalent if there exists a G-fibration $q : Z \to B \times I$ such that $q|_{B\times\{0\}} = p_1$ and $q|_{B\times\{1\}} = p_2$. We write $p_1 \simeq_s p_2$. The G-fibration q is called a G-connection.*

It is well-known that a G-fiber homotopy equivalence is an equivalence relation. In [24], it is shown that the property of being strong fiber homotopy equivalent is an equivalence relation. The same is also true for strong G-fiber homotopy equivalence.

Proposition 2.4.1 *Being a strongly G-fiber homotopy equivalent is an equivalence relation.*

Proof: It is clear that the property of being strong G-fiber homotopy equivalent is reflexive and symmetric. To prove the transitivity, let $p_i : E_i \to B$ be G-fibrations for $i = 1, 2, 3$ such that $p_1 \simeq_s p_2$ and $p_2 \simeq_s p_3$. Let $p_{12} : Z_1 \to B \times I$ and $p_{23} : Z_2 \to B \times I$ be G-connections between p_1 and p_2, p_2 and p_3 respectively. Note that $p_{12}^{-1}(B \times \{1\}) = p_{23}^{-1}(B \times \{0\})$. Let

$$Z = Z_1 \cup_{i_1} E_2 \times I \cup_{i_2} Z_2$$

where $i_1(e) = (e, 0)$ and $i_2(e) = (e, 1)$ for every $e \in E_2$. By uniformization theorem $q : Z \to B \times I$ given by

$$q(z) = \begin{cases} (\pi_1(p_{12}(z)),\ \frac{1}{3}\pi_2(p_{12}(z))), & z \in Z_1; \\ (\pi_1(p_{23}(z)),\ \frac{2}{3} + \frac{1}{3}\pi_2(p_{23}(z))), & z \in Z_2; \\ (p_2(e), \frac{1}{3}(1+t)), & e \in E_2. \end{cases}$$

is a G-fibration with $q|_{(B\times\{0\})} = p_1$ and $q|_{(B\times\{1\}} = p_3$. $\square$

We refer the reader [14] for the proof of non-equivariant version of the following observation.

Proposition 2.4.2 *Let $p_i : E_i \to B$ be strong G-fiber homotopy equivalent G-fibrations for $i = 1, 2$. If $f, g : X \to B$ are G-homotopic then the induced G-fibrations $f^*(p_1)$ and $f^*(p_2)$ are strong G-fiber homotopy equivalent.*

Proposition 2.4.3 *Let p_1 and p_2 be strongly G-fiber homotopy equivalent G-fibrations. Then they are G-fiber homotopy equivalent.*

Proof: Let $q : Z \to B \times I$ be a G-connection between p_1 and p_2. Since the followings are commutative diagrams of G-maps

$$\begin{tikzcd} E_1 \times \{0\} \arrow[r, hook] \arrow[d, hook] & Z \arrow[d, "q"] \\ E_1 \times I \arrow[r, "p_1 \times \mathrm{id}"'] & B \times I \end{tikzcd} \qquad \begin{tikzcd} E_2 \times \{0\} \arrow[r, hook] \arrow[d, hook] & Z \arrow[d, "q"] \\ E_2 \times I \arrow[r, "p_2 \times (1 - \mathrm{id})"'] & B \times I \end{tikzcd}$$

there are G-maps $F : E_1 \times I \to Z$ and $G : E_2 \times I \to Z$ which make the above diagrams commute. Then $f = F|_{E_1 \times \{1\}}$ is a desired G-fiber homotopy equivalence with G-fiber homotopy inverse $g = G|_{E_2 \times \{1\}}$. More precisely, if we define

$$h : E_1 \times I \times \{0, 1\} \cup E_1 \times \{0\} \times I \to Z$$

by

$$h(e, s, t) = \begin{cases} e, & t = 1 \text{ or } s = 0; \\ F(e, 2s), & t = 0,\ 0 \le s \le \frac{1}{2}, \\ G(f(e), 2s - 1), & t = 0,\ \frac{1}{2} \le s \le 1. \end{cases}$$

then $p \circ h = H(-, -, 0)$ where $H(e, s, t) = (p_1(e), 0)$. Therefore there is a G-map $\widetilde{H} : E_1 \times I \times I \to Z$ such that $p\widetilde{H} = H$ and $\widetilde{H}|_{E_1 \times I \times \{0,1\} \cup E_1 \times \{0\} \times I} = h$. Then $\bar{H} = \widetilde{H}|_{E_1 \times \{1\} \times I} : E_1 \times I \to E_1$ is a G-homotopy between $\widetilde{H}(-, 1, 1) = \mathrm{id}_{E_1}$ and $\widetilde{H}(-, 1, 0) = g \circ f$. Similarly, one can obtain a G-homotopy between $f \circ g$ and id_{E_2}. $\square$

2.5 Tulley's theorem for G-fibrations

In [25], Tulley shows that two fibrations are fiber homotopy equivalent if and only if they are strongly fiber homotopy equivalent. In this section, we prove the following generalization of this result by using the same methods and ideas from [24] and [25].

Theorem 2.5.1 *Let $p_i : E_i \to B$ be G-fibration for $i = 1, 2$. Then p_1 and p_2 are G-fiber homotopy equivalent if and only if they are strongly G-fiber homotopy equivalent.*

One direction of the above theorem is proved in the previous section. For the other direction we need to consider the mapping cylinder of the G-homotopy equivalence $f : E_1 \to E_2$. For an arbitrary fiber preserving G-map $f : E_1 \to E_2$, the mapping cylinder of f yields a G-map $p_f : M_f \to B$ over B which is defined by $p_f(x, s) = p_1(x)$ and $p_f(y) = p_2(y)$ for any $x \in E_1$ and $y \in E_2$ and $s \in I$.

Lemma 2.5.1 *Let $p_i : E_i \to B$ be G-fibrations for $i = 1, 2$. If $f : p_1 \to p_2$ is a G-fiber map then $p_f : M_f \to B$ is a G-fibration.*

Proof: Let $\lambda_i : \Omega_{p_i} \to {E_i}^I$ be a G-lifting function for p_i, $i = 1, 2$. Note that $\Omega_{p_f} = \Omega_{p_1} \times I \cup_{\widetilde{f}} \Omega_{p_2}$ where $\widetilde{f}((e, \omega), 0) = (f(e), \omega)$. Therefore it suffices to construct a G-map $\lambda : \Omega_{p_1} \times I \to (M_f)^I$ such that $\lambda|_{\Omega_{p_1} \times \{0\}} = \lambda_2(\widetilde{f})$ and $p_f \lambda((e, \omega), s)(t) = \omega(t)$.

Let $\widetilde{p_1} : E_1 \times I \to B$ be the G-fibration given by $\widetilde{p_1}(e, t) = p_1(x)$. Define $r : E_1 \times I \times I \to E_1 \times I \times \{0\} \cup E_1 \times \{0\} \times I$ by

$$r(e, s, t) = \begin{cases} (e, 0, 1 - 2s), & s + t \geq 1,\ s \leq \frac{1}{2}; \\ (e, 2s - 1, 0), & s + t \geq 1,\ s \geq \frac{1}{2}; \\ (e, 0, t - s), & s + t \leq 1,\ s \leq t; \\ (e, s - t, 0), & s + t \leq 1,\ s \geq t. \end{cases}$$

Here r is a fiber preserving G-retraction in the sense that $\widetilde{p_1}\pi_1 r(e, s, t) = p_1(e)$ where $\pi_1 : E_1 \times I \times \{0\} \cup E_1 \times \{0\} \times I \to E_1 \times I$ is the projection. Let us now define $\lambda : \Omega_p \times I \to M_f^I$ by

$$\lambda(e,\omega,s)(t) = \begin{cases} \pi_1 r(\lambda_1(e,\omega)(t), s, t), & 0 \le t \le 1,\ s \ge \frac{1}{2}; \\ \pi_1 r(\lambda_1(e,\omega)(t), s, \frac{t}{2s}) & 0 \le t \le 2s,\ 0 < s \le \frac{1}{2}; \\ \lambda_2(f(\lambda_1(e,\omega)(2s)), \omega^{2s})(t-2s), & 0 \le 2s \le t \le 1. \end{cases}$$

where ω^s is given by the equation 2.1. Since r is a fiber preserving G-map and λ_1 and λ_2 are G-lifting functions, λ is a G-map satisfying the relation $p_f\lambda((e,\omega),s)(t) = \omega(t)$. On the other hand, when $s = 0$, we have

$$\lambda(e,\omega,0)(t) = \lambda_2(f(\lambda_1(e,\omega)(0)), \omega^0)(t) = \lambda_2(f(e),\omega)(t).$$

Therefore we only need to check the continuity of λ. For this, we show that the adjoint function $\widetilde{\lambda} : \Omega_{p_1} \times I \times I \to M_f$ given by $\widetilde{\lambda}(e,\omega,s,t) = \lambda(e,\omega,s)(t)$ is continuous.

Clearly the restrictions of $\widetilde{\lambda}$ to the closed subsets

$$C_1 = \{(e,\omega,s,t) \in \Omega_{p_1} \times I \times I |\ 0 \le t \le 1 \text{ and } s \ge \frac{1}{2}\}$$

and

$$C_2 = \{(e,\omega,s,t) \in \Omega_{p_1} \times I \times I |\ 0 \le 2s \le t \le 1\}$$

are continuous. Therefore it suffices to show the continuity of $\widetilde{\lambda}|_{C_3}$ where

$$C_3 = \{(e,\omega,s,t) \in \Omega_{p_1} \times I \times I |\ 0 \le t \le 2s \text{ and} 0 \le s \le \frac{1}{2}\}.$$

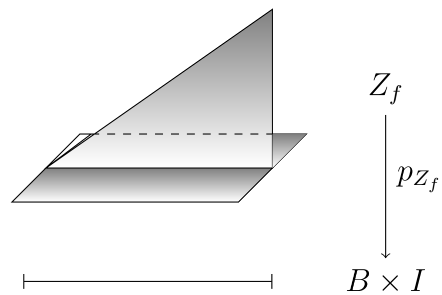
However the restriction of $\widetilde{\lambda}|_{C_3}$ is given for any $(e,\omega,s,t) \in C_3$ by

$$\widetilde{\lambda}|_{C_3}(e,\omega,s,t) = \begin{cases} (\lambda_1(e,\omega)(t), s - \frac{t}{2s}), & \frac{t}{2s} \le s; \\ (\lambda_1(e,\omega)(t), 0), & \text{otherwise.} \end{cases}$$

and hence it is continuous. $\square$

Since a strong G-fiber homotopy equivalence is an equivalence relation, Theorem 2.5.1 follows if we show that each of the fibrations p_1 and p_2 are G-strongly fiber homotopy equivalent to p_f when f is a G-fiber homotopy equivalence. The latter holds for arbitrary G-fiber maps.

Proposition 2.5.1 *Let* $Z_f = \{(e,s,t) \in E_1 \times I \times I|\ s \leq t\} \cup_{\widetilde{f}} E_2 \times I \subset M_f \times I$ *where* $\widetilde{f} : E_1 \times \{0\} \times I$ *is defined by* $\widetilde{f}(e,0,t) = (f(e),t)$*. Then* $p_{Z_f} : Z_f \to B \times I$ *where* $p_{Z_f} = (p_f \times \mathrm{id})|_{Z_f}$ *is a* G*-fibration. In particular,* p_2 *and* M_f *are* G*-strongly fiber equivalent.*



Proof: Let $r : M_f \times I \to Z_f$ be defined by

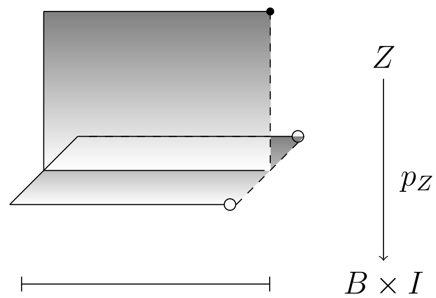
$$r|_{E_1 \times I \times I}(x,s,t) = \begin{cases} (x,s,t) & s \leq t; \\ (x,t,t), & \text{otherwise.} \end{cases} \tag{2.2}$$

and $r|_{E_2} = \mathrm{id}_{E_2}$. Then r is a fiber preserving G-retraction. Therefore p_{Z_f} is a G-fibration by Proposition 2.1.2. □

Definition 2.5.1 *Let* $p_i : E_i \to B$ *be a* G*-fibration for* $i = 1,2$ *with* $E_1 \subseteq E_2$ *and* $p_2|_{E_1} = p_1$*. Then* p_1 *is said to be a* G*-deformation retract of* p_2 *if* E_1 *is a* G*-deformation retract of* E_2 *via fiber preserving* G*-retraction, that is, if there is a* G*-map* $H : E_2 \times I \to E_2$ *such that* $H_0 = \mathrm{id}_{E_2}$*,* $H(e,1) \in E_1$ *and* $p_2 H(e,t) = p_2(e)$ *for every* $e \in E_2$ *. If* H *also satisfies the relation* $H(e,t) = e$ *for every* $e \in E_1$*, we say* p_1 *is a strong* G*-deformation retract of* p_2*.*

To show that p_1 and p_f are strongly G-fiber homotopy equivalent, we need the special case of Theorem 2.5.1 where p_1 is assumed to be G-fiber strong deformation retract of p_2. The non-equivariant version of this special case is proved in [24]. This result also used in a recent paper by Steimle [22].

Proposition 2.5.2 *If* p_1 *is a strong* G*-deformation retract of* p_2 *then they are strong* G*-fiber homotopy equivalent via the* G*-connection* $(p_2 \times \mathrm{id})|_Z : Z \to B$ *where* $Z = \{(e,t) \in E_2 \times I|\ e \in E_2 \text{ if } t > 0,\ e \in E_1 \text{ if } t = 0\}$*.*



Proof: Let $H : E_2 \times I \to E_2$ be a G-map such that $H(e,0) = e$, $H(e,1) \in E_1$ and $p_2 H(e,t) = p_2(e)$ for every $e \in E_2$ and $H(e,t) = e$ for every $e \in E_1$. Let $\lambda : \Omega_{p_2} \to E_2^I$ be a G-lifting function for p_2. Define a G-map $\lambda' : \Omega_q \to Z^I$ by $\pi_2 \lambda'((e, \omega_2(0)), (\omega_1, \omega_2))(t) = \omega_2(t)$ and

$$\pi_1 \lambda'((e, \omega_2(0)), (\omega_1, \omega_2))(t) = \begin{cases} H(\lambda(e, \omega_1)(t)), \frac{t}{\omega_2(t)}), & \omega_2(t) > 0,\ \omega_2(t) \geq t; \\ e, & t = \omega_2(t); \\ H(\lambda(e, \omega_1)(t)), 1), & t > 0,\ t \geq \omega_2(t). \end{cases}$$

Clearly, $p\lambda'((e, \omega_2(0)), (\omega_1, \omega_2)) = (\omega_1, \omega_2)$ and $\lambda'((e, \omega_2(0)), (\omega_1, \omega_2))(0) = e$. Therefore we only need to check continuity of $\pi_1 \lambda'$ at $t = 0$. For this it suffices to show that the adjoint map $\widetilde{\pi_1 \lambda'} : \omega_q \times I \to E_2$ is continuous at $t = 0$.

Let $(e_\alpha, \omega_{1,\alpha}, \omega_{2,\alpha}, t_\alpha)$ be a net converging to $(e, \omega_1, \omega_2, 0)$. Let U be an open neighborhood of $e \in E_1$. Since $H : E_2 \times I \to E_2$ is continuous, $V = H^{-1}(U)$ is open. Since $(e,t) \in V$ for every $t \in I$, there are open neighborhoods $A_t \ni e$ and $V_t \ni t$ such that $A_t \times V_t \subseteq V$, for all $t \in I$. Since I is compact, there exist $t_1, \ldots, t_n$ such that $I = \cup_{i=1}^n V_{t_i}$. Then $A = \cap_{i=0}^n A_{t_i}$ is an open neighborhood of e with the property that $H(A \times I) \subseteq U$. Since λ is continuous, there is β such that $\widetilde{\lambda}(e_\alpha, \omega_{1,\alpha}, t_\alpha) \in A$ for every $\beta > \alpha$. Therefore $\widetilde{\pi_1 \lambda'}(e_\alpha, \omega_{1,\alpha}, \omega_{2,\alpha}, t_\alpha) \in U$ for every $\alpha > \beta$ as desired. $\square$

In the light of the above proposition, it suffices to show that p_1 is a strong G-deformation retract of p_f in order to prove Theorem 2.5.1. The standard way of proving that the total space E_1 of p_1 is a strong G-deformation retract of the total space M_f of p_f is to show that the pair (E_2, E_1) is a G-cofibration and there is a G-deformation retraction of M_f onto E_1. Here, we also use the same method.

Definition 2.5.2 *Let $p_i : E_i \to B$ be G-fibration for $i = 1, 2$ with $E_1 \subseteq E_2$ and $p_2|_{E_1} = p_2$. A pair (p_2, p_1) is said to be a G-cofibration if there is a G-retraction $r : E_2 \times I \to E_2 \times \{0\} \cup E_1 \times I$ such that $p_2\pi_1 r(e,t) = p_2(e)$ for every $e \in E_2$ and $t \in I$.*

Note that $r : M_f \times I \to M_f \times \{0\} \cup E_1 \times \{1\} \times I$ defined by

$$r(x,s,t) = \begin{cases} (x, s+t, 0), & t \leq s \text{ and } 0 \leq s+t \leq 1; \\ (x, 1, s+t-1), & t \leq s \text{ and } 1 \leq s+t \leq 2; \\ (x, 2s, 0), & t \geq s \text{ and } s \leq \frac{1}{2}; \\ (x, 1, 2s-1), & t \geq s \text{ and } s \geq \frac{1}{2}. \end{cases}$$

for every $(x,s,t) \in E_1 \times I \times I$ and $r(y,t) = y$ for every $(y,t) \in E_2 \times I$ is a G-retraction with $p_2\pi_1 r(e,t) = p_2(e)$ for all $(e,t) \in M_f \times I$. Therefore (p_f, p_1) is a G-cofibration.

It is well-known that E_1 is a G-deformation retract of M_f when f is a G-homotopy equivalence. Via the same homotopy, p_1 is a G-deformation retract of p_f. More precisely, let $H_i : E_i \times I \to E_i$ be fiber preserving G-homotopies with $H_1(-,0) = gf$, $H_1(-,1) = \mathrm{id}_X$, $H_2(-,0) = fg$ and $H_2(-,1) = \mathrm{id}_Y$. Now define $H : M_f \times I \to M_f$ by

$$H(x,s,t) = \begin{cases} (x, s(1-3t)), & 0 \leq t \leq \frac{1}{3}; \\ H_2(f(x), 2-3t), & \frac{1}{3} \leq t \leq \frac{2}{3}; \\ (H_1(x, (3t-2)s), 3t-2), & \frac{2}{3} \leq t \leq 1. \end{cases}$$

for every $(x,s,t) \in E_1 \times I \times I$ and for every $(y,t) \in E_2 \times I$

$$H(y,t) = \begin{cases} y, & 0 \leq t \leq \frac{1}{3}; \\ H_2(y, 2-3t), & \frac{1}{3} \leq t \leq \frac{2}{3}; \\ (g(y), 3t-2), & \frac{2}{3} \leq t \leq 1. \end{cases}$$

Since H_1, H_2, f and g are fiber preserving G-maps so is H. Also $H(-,0)$ is identity on M_f and

$$H(x,s,1) = (H_1(x,s), 1) \in E_1 \times \{1\},\ H(y,1) = (g(y), 1) \in E_1 \times \{1\},$$

that is, $H_{M_f \times \{1\}}$ is a G-retraction and hence H is a G-deformation retraction of M_f onto E_1. Unfortunately it is not a strong G-deformation retract. However, when a G-cofibration is a G-deformation retract it is a strong G-deformation retraction by the following lemma which is an analogous of a standard result in homotopy theory. We refer reader to [19] for the proof of this standard result since the same proof applies here.

Lemma 2.5.2 *If (p_2, p_1) is a G-cofibration and p_1 is a G-deformation retract of p_2 then p_1 is a strong G-deformation retract of p_2.*

Therefore, by Proposition 2.5.2 p_1 is strongly G-fiber homotopy equivalent to p_f and by transitivity property, p_1 and p_2 are strong G-fiber homotopy equivalent. Now we can prove the following analogue of Corollary 2.2.1.

Corollary 2.5.1 *Let $B \subseteq B_1 \cap B_2$. If $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ are G-fibrations with $p_1|_B \simeq p_2|_B$ then there is a G-fibration over $B_1 \cup_{i_1} B \times I \cup_{i_2} B_2$ extending p_1 and p_2 where $i_j : B \times \{j\} \to B_j$ are inclusions.*

Proof: By Theorem 2.5.1, G-fibrations $p_1|_B$ and $p_2|_B$ are strong G-fiber homotopy equivalent. Let $q : Z \to B \times [\frac{1}{3}, \frac{2}{3}]$ be the G-connection between $p_1|_B$ and $p_2|_B$. Note that there is no loss of generality in taking G-connection over $B \times [\frac{1}{3}, \frac{2}{3}]$ with $q|_{B\times\{\frac{1}{3}\}} = p_1$ and $q|_{B\times\{\frac{2}{3}\}} = p_2$.

Let

$$\widetilde{Z} = E_1 \cup_{k_1} p_1^{-1}(B) \times [0, \frac{1}{3}] \cup_{m_1} Z \cup_{m_2} p_2^{-1}(B) \times [\frac{2}{3}, 1] \cup_{k_2} E_2$$

where $k_j : p_1^{-1}(B) \times \{j-1\} \to E_j$ and $m_j : p_j^{-1}(B) \times \{\frac{j}{3} \to Z$ are the inclusions for $j = 1, 2$. Define a G-map $\widetilde{q} : \widetilde{Z} \to B_1 \cup_{i_1} B \times I \cup_{i_2} B_2$ by $\widetilde{q}|_{E_i} = p_i$, $\widetilde{q}|_{Z\times[\frac{1}{3},\frac{2}{3}]}(z, t) = q(z)$ and by $\widetilde{q}(e, t) = p_j(e)$ for $(e, t) \in p_j^{-1}(B \times [\frac{2(j-1)}{3}, \frac{2j-1}{3}])$. Clearly the restriction of $\widetilde{q}$ to the following subsets are G-fibrations

$$\{B_1 \cup_{i_1} B \times [0, \frac{2}{9}], B \times [\frac{1}{9}, \frac{5}{9}], B \times [\frac{4}{9}, \frac{8}{9}], B \times [\frac{7}{9}, 1] \cup_{i_2} E_2\}$$

Therefore, $\widetilde{q}$ is a G-fibration by uniformization theorem. $\square$

Recall that a G-CW-complex is the colimit of a sequence of G-inclusions

$$X^0 \subset X^1 \subset \cdots \subset X^{n-1} \subset X^n \subset \cdots,$$

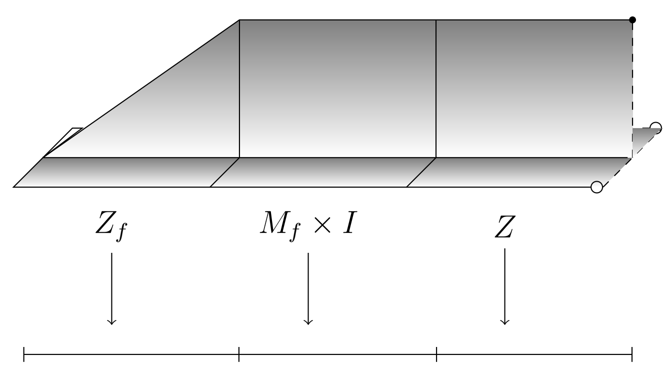
where X^n is obtained from X^{n-1} by attaching disjoint union of equivariant n-dimensional cells, that is, there exists a G-pushout diagram

$$\begin{tikzcd} \coprod_{\sigma\in I_n} G/H_\sigma \times \mathbb{S}^{n-1} \arrow[r] \arrow[d, hook] & X^{n-1} \arrow[d] \\ \coprod_{\sigma\in I_n} G/H_\sigma \times \mathbb{D}^{n-1} \arrow[r] & X^n. \end{tikzcd}$$

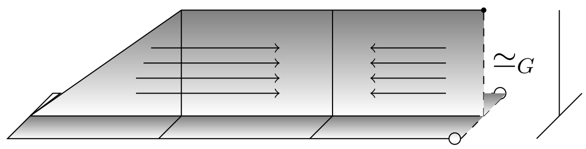
Theorem 2.5.2 *Let $p_i : E_i \to B$ be G-fiber homotopy equivariant G-fibrations such that E_1 and E_2 have the G-homotopy type of a G-CW-complex. Then there is a G-connection between them with total space which is G-homotopy equivalent to a G-CW-complex.*

Proof: Let $f : E_1 \to E_2$ be a G-fiber homotopy equivalence with inverse $g : E_2 \to E_1$. In the previous section, we prove that $q : Z' \to B$ is a G-connection between p_1 and p_2 where $Z' = Z_f \cup_{i_1} M_f \times [\frac{1}{3}, \frac{2}{3}] \cup Z$. Here

$$\begin{aligned} Z_f &= \{(x,s,t) \in E_1 \times I \times [0, \frac{1}{3}] |\ s \leq 3t\} \cup_f E_2 \times [0, \frac{1}{3}] \\ Z &= \{(e,t) \in E_2 \times [\frac{2}{3}, 1] |\ e \in E_2 \text{ if } t < 1,\ e \in E_1 \text{ if } t = 1\}. \end{aligned}$$

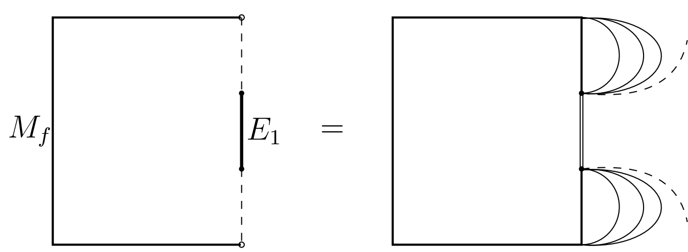


To prove the theorem, it suffices to show that Z' has a G-homotopy type of a finite G-CW-complex. For this first note that Z is a strong G-deformation retract of M_f.



On the other hand, M_f is G-homotopic to E_2 and hence it has a G-homotopy type of a G-CW-complex.

Remark 2.5.1 *In the previous theorem, we can replace the conditions of being G-homotopy type of a G-CW-complex to that of being actual G-CW-complexes. Unfortunately, in this case, the total space of the G-connection will be infinite G-CW-complex not a finite one. The only problem here is that the space Z does not have a finite G-CW-complex structure since it is not compact. The CW-complex structure of Z is obtained by attaching infinitely many cells to $M_f \times I$ as shown in the picture.*



Corollary 2.5.2 *A G-fibration $p : E \longrightarrow \mathbb{S}^{n-1}$ over $(n-1)$-sphere with fiber F_G extends to a G-fibration over disk if and only if it is G-fiber homotopy equivalent to a trivial G-fibration.*

Proof: Since $\mathbb{D}^n$ is contractible, only if part holds. For the other direction, let $q : Z \longrightarrow \mathbb{S}^{n-1} \times [0, \frac{1}{3}]$ be the G-connection between p and the trivial G-fibration. Now consider $\mathbb{D}^n$ as the cone of $\mathbb{S}^{n-1}$. Divide it into two parts as follows. Let $B_1 = \mathbb{S}^{n-1} \times [0, \frac{1}{3}]$ and $B_2 = \mathbb{S}^{n-1} \times [\frac{1}{3}, 1] \backslash (x, 1) \simeq *$. Let $CE = Z \cup_i B_2 \times F_G$ where $i = \mathrm{id}_{\mathbb{S}^{n-1} \times \{\frac{1}{3}\} \times F}$. Let $Cp : CE \longrightarrow \mathbb{D}^n$ be the G-map defined by $Cp|_{B_1} = q$, $Cp(x, t, y) = (x, t)$ for $x \in \mathbb{S}^{n-1}$, $t \in [\frac{1}{3}, 1]$ and $y \in F_G$. Clearly, $Cp|_{\mathbb{S}^{n-1} \times [0, \frac{2}{3}]}$ and $Cp|_{\mathbb{S}^{n-1} \times [\frac{1}{3}, \frac{2}{3}]}$ are G-fibrations. Since $\{\mathbb{S}^{n-1} \times [0, \frac{2}{3}],\ \mathbb{S}^{n-1} \times [\frac{1}{3}, \frac{2}{3}]\}$ is a numerable covering of $\mathbb{D}^n$, Cp is a G-fibration by uniformization theorem. $\square$

Chapter 3

Constructing G-fibrations

The aim of this chapter is to develop an obstruction theory for constructing G-fibrations. An adequate cohomology theory for this obstruction theory is Bredon cohomology. This cohomology theory is established by Bredon [4] in order to develop an equivariant obstruction theory for extending G-maps between G-spaces. We begin this chapter with a brief introduction to Bredon cohomology.

In the second section, we discuss the equivariant obstruction theory for extending G-maps between G-spaces. This theory works as the same way the nonequivariant one does. Here the strategy is to construct the map cell-by-cell. In the last section, we will use the same strategy to define obstructions for constructing a G-fibration over a given G-CW-complex.

3.1 Bredon Cohomology

Let G be a finite group and $\mathcal{H}$ be a family of subgroups of G which is closed under conjugation and taking subgroups.

Definition 3.1.1 *The orbit category* $\mathrm{Or}_{\mathcal{H}}(G)$ *of* G *is the category whose objects are the left cosets* G/H *where* $H \in \mathcal{H}$ *and whose morphisms are the* G*-maps from*

G/H to G/K.

Note that there exists a G-map $f : G/H \to G/K$ if and only if $H^a \leq K$. More precisely, if $f : G/H \to G/K$ is a G-map with $f(H) = aK$ then for any $h \in H$,

$$aK = f(H) = f(hH) = hf(H) = haK, \text{ i.e. } a^{-1}ha \in K$$

Conversely, any $a \in G$ such that $H^a \leq K$ defines a G-map $f : G/H \to G/K$ given by $f(H) = aK$. From now on we denote any such f by $\hat{a}$.

Definition 3.1.2 *A coefficient system for Bredon cohomology is a contravariant functor $M : \mathrm{Or}_{\mathcal{H}}(G) \to \mathcal{A}b$ where $\mathcal{A}b$ is the category of abelian groups.*

A morphism $T : M \to N$ between coefficient systems is a natural transformation of functors. Note that a coefficient system is a $\mathbb{Z}\mathrm{Or}_{\mathcal{H}}(G)$-module with the usual definition of modules over a small category. The $\mathbb{Z}\mathrm{Or}_{\mathcal{H}}(G)$-module category is an abelian category, so the usual notions for doing homological algebra exist.

Let (X, A) be a relative G-CW-complex whose all isotropy subgroups in $\mathcal{H}$. In this section, we often use the following coefficient systems.

Example 3.1.1 *If (X, A) is a G-CW-complex then any $\hat{a} \in \hom(G/H, G/K)$ induces a map*

$$\bar{a} : (X^K, A^K) \to (X^H, A^H)$$

given by left multiplication by a. Define $\underline{C}_n(X, A) : \mathrm{Or}_{\mathcal{H}}(G) \to \mathcal{A}b$ by

$$\underline{C}_n(X, A)(G/H) = C_n(X^H, A^H; \mathbb{Z})$$

$$\underline{C}_n(X, A)(\hat{a}) = \bar{a}_* : C_n(X^K, A^K; \mathbb{Z}) \to C_n(X^H, A^H; \mathbb{Z})$$

for any $H, K \in \mathcal{H}$ and $a \in G$ such that $H^a \leq K$.

Example 3.1.2 *Similar to Example 3.1.1, we can define $\underline{\pi}_n(X, A) : \mathrm{Or}_{\mathcal{H}}(G) \to \mathcal{A}b$ by*

$$\underline{\pi}_n(X,A)(G/H) = \pi_n(X^H, A^H)$$

$$\underline{\pi}_n(X,A)(\hat{a}) = \bar{a}_* : \pi_n(X^K, A^K) \to \pi_n(X^H, A^H)$$

for any $H, K \in \mathcal{H}$ and $a \in G$ such that $H^a \leq K$.

To simplify the notation, we write $M(H)$ and $f(H)$ for $M(G/H)$ and $f(G/H)$ respectively. We also denote $\underline{\pi}_n(X,A)$ by $\underline{\pi}_n$ and $\underline{C}_n(X,A)$ by $\underline{C}_n$ when it is clear from the context that which relative pair we are working with.

Given a local coefficient system $M : \mathrm{Or}_{\mathcal{H}}(G) \to \mathcal{A}b$, one defines the cochain complex $\underline{C}^*(X,A;M)$ of (X,A) with coefficients in M as follows: Let $\underline{C}^n(X,A;M) = \mathrm{Hom}_{\mathrm{Or}_{\mathcal{H}}(G)}(C_n;M)$ which is the submodule of $\oplus_{H\in\mathcal{H}}\mathrm{Hom}_{\mathbb{Z}N_G(H)/H}(C_n(X^H,A^H;\mathbb{Z}), M(H))$ formed by elements $(f(H))_{H\in\mathcal{H}}$ such that the following diagram commutes:

$$\begin{array}{ccc} C_n(X^K, A^K; \mathbb{Z}) & \xrightarrow{f(K)} & M(K) \\ {\scriptstyle \bar{a}_*}\big\downarrow & & \big\downarrow{\scriptstyle M(\hat{a})} \\ C_n(X^H, A^H\mathbb{Z}) & \xrightarrow{f(H)} & M(H) \end{array}$$

for any $H, K \in \mathcal{H}$ and for any $a \in G$ such that $H^a \in K$. The coboundary map $\delta : \underline{C}^n(X,A;M) \to \underline{C}^{n+1}(X,A;M)$ is defined by $(\delta f)(H)(\tau) = f(H)(\partial\tau)$ for any $H \in \mathcal{H}$ and for any $(n+1)$-cell τ of (X^H, A^H).

Definition 3.1.3 *Let (X,A) be a relative G-CW-complex and M be a $\mathbb{Z}Or_{\mathbb{H}}(G)$-module. The Bredon cohomology $H^*_G(X,A;M)$ of (X,A) with coefficients in M is defined as the cohomology of the cochain complex $\underline{C}^*(X,A;M)$.*

Remark 3.1.1 *If X is a G-CW-complex whose all isotropy subgroups are in $\mathcal{H}$, then $\underline{C}_n(X) = \oplus_{\sigma\in I_n}\mathbb{Z}[G/G_\sigma]$ where I_n is the set of orbit representatives of n-cells of X. Therefore we have*

$$\mathrm{Hom}_{\mathrm{Or}_{\mathcal{H}}(G)}(\underline{C}_n; M) \cong \bigoplus_{\sigma\in I_n} M(G_\sigma)$$

by Yoneda lemma.

3.2 Equivariant Obstruction Theory

In this section we discuss the equivariant obstruction theory developed by Bredon. We refer the reader to [4] and [7] for more details.

Let (X, A) be a relative G-CW-complex, $n \geq 1$ and let Y be a G-space such that $Y^H = \{y \in Y | \ hy = y, \ \forall \ h \in H\}$ is non-empty, path-connected and n-simple for every isotropy subgroup H of G-action on X. Recall that a path-connected space Z is said to be n-simple if $\pi_1(Z)$ acts trivially on $\pi_q(Z)$ for $q \leq n$. Given an equivariant map $\phi : X^n \cup A \to Y$, define an element c_ϕ in $\oplus_{H \in \mathcal{H}} \mathrm{Hom}_{\mathbb{Z}N_G(H)/H}(C_{n+1}(X^H, A^H; \mathbb{Z}), \pi_n(Y)(H))$ by

$$c_\phi(H)(\sigma) = [\phi \circ f_\sigma]$$

where $f_\sigma : \mathbb{S}^n \to X^n$ is the attaching map for $\sigma \in C_{n+1}(X^H, A^H)$. Here $\mathrm{Im}(\phi \circ f_\sigma) \subseteq Y^H$ since ϕ is a G-map and $H \leq G_\sigma$. If $a \in G$ such that $H^a \leq K$ and σ is an $(n+1)$-cell of (X^K, A^K) with characteristic map f_σ then $a\sigma$ is an $(n+1)$-cell of (X^H, A^H) and the attaching map of $a\sigma$ is given by $f_{a\sigma} = af_\sigma$. Therefore the following diagram commutes

$$\begin{array}{ccc} C_n(X^K, A^K; \mathbb{Z}) & \xrightarrow{c_\phi(K)} & \pi_n(Y^K) \\ {\scriptstyle \tilde{a}_*}\Big\downarrow & & {\scriptstyle M(\bar{a})}\Big\downarrow \\ C_n(X^H, A^H; \mathbb{Z}) & \xrightarrow{c_\phi(H)} & \pi_n(Y^H) \end{array} \tag{3.1}$$

and hence c_ϕ is an element of $C^{n+1}(X, A; \underline{\pi})$. The cochain c_ϕ is called the obstruction cochain. As in the non-equivariant obstruction theory, c_ϕ has the following property.

Proposition 3.2.1 *The obstruction cochain c_ϕ is a cocycle.*

Proof: Let $H \in \mathcal{H}$ and let τ be an $(n+2)$-cell of X^H. Note that

$$(\delta c_\phi)(H)(\tau) = c_\phi(H)(\partial\tau) = [\phi \circ f_{\partial\tau}] = (\delta o|_{\phi'})(\tau)$$

where $o|_{\phi'}$ is the classical obstruction to extending the restriction $\phi' = \phi|_{X^H \cup A^H}$. Then by non-equivariant obstruction theory, $(\delta c_\phi)(H)(\tau) = 0$ for every $H \in \mathcal{H}$ and $\tau \in C_{n+2}(X^H, A^H)$ and hence $\delta c_\phi = 0$. $\square$

Proposition 3.2.2 *A G-map $\phi : X^n \cup A \to Y$ can be extended to $X^{n+1} \cup A$ if and only if $c_\phi = 0$.*

Proof: Let I_{n+1} be the set of representatives of G-orbits of $(n+1)-$cells. If $c_\phi = 0$, then $c_\phi(G_\sigma)(\sigma) = 0$. Therefore by standard obstruction theory, we can extend ϕ to $X^n \cup A \cup \sigma$ for each $\sigma \in I_{n+1}$ in a such a way that $\phi(\sigma) \in Y^{G_\sigma}$. Then we can extend ϕ to $X^{n+1} \cup A$ by letting $\phi(gx) = g\phi(x)$ for every $x \in g\sigma$ for some $\sigma \in I_{n+1}$ and for some $g \in G$. $\square$

Since c_ϕ is a cocyle, it represents a cohomology class. In non-equivariant theory, the cohomology class representing the obstruction cocyle vanishes if and only if one can extend the map by redefining it on the n-skeleton. In order to obtain the analogues result for the equivariant case, the difference cochain is defined similarly as follows. Let ϕ and θ be G-maps from $X^n \cup A$ to Y such that their restrictions to $X^{n-1} \cup A$ are G-homotopy equivalent. Let $F : (X^{n-1} \cup A) \times I \to Y$ be a G-homotopy between them. Let $c_{\phi,F,\theta}$ be the obstruction cocycle to extending the map $\widetilde{F}$ equivariantly where

$$\widetilde{F} : (X \times I)^n \cup A \times I \to Y$$

is the G-map defined by the relations $\widetilde{F}|_{(X^{n-1} \cup A) \times I} = F$, $\widetilde{F}|_{X^n \times \{0\}} = \phi$ and $\widetilde{F}|_{X^n \times \{1\}} = \theta$. Then the *difference cochain* $d_{\phi,F,\theta} \in C^n_G(X, A; \underline{\pi})$ is defined by

$$d_{\phi,F,\theta}(H)(\sigma) = (-1)^{n+1} c_{\widetilde{F}}(H)(\sigma \times I)$$

for every $H \leq G$ and for every n-cell σ of X^H.

Lemma 3.2.1 *$\delta d_{\phi,F,\theta} = c_\phi - c_\theta$.*

Proof: Since $c_{\widetilde{F}}(\sigma \times I)$ is a cocycle, for every $H \leq G$ and every $(n+1)$-cell τ of X^H, we have

$$\begin{aligned} 0 &= \delta c_{\widetilde{F}}(H)(\tau \times I) = c_{\widetilde{F}}(H)(\partial\tau \times I) + (-1)^{n+1}(c_{\widetilde{F}}(H)(\tau \times 1) - c_{\widetilde{F}}(H)(\tau \times 0)) \\ &= (-1)^{n+1}(\delta d_{\phi,F,\theta}(H)(\tau) + c_\theta(H)(\tau) - c_\phi(H)(\tau)) \end{aligned}$$

i.e. $\delta d_{\phi,F,\theta}(H)(\tau) = c_\phi(H)(\tau) - c_\theta(H)(\tau)$ as desired. $\square$

Proposition 3.2.3 *Given a G-map $\phi : X^n \cup A \to Y$ and a cochain $d \in C^n(X, A; \underline{\pi})$, there is a G-map $\theta : X^n \cup A \to Y$ such that $\phi|_{X^{n-1}\cup A} = \theta|_{X^{n-1}\cup A}$ and $d = d_{\phi,\mathrm{Id},\theta}$.*

Proof: For any $\sigma \in (X, A)^n$, let $\Phi : \mathbb{D}^n \times \{0\} \cup \mathbb{S}^{n-1} \times I \to Y^{G_\sigma}$ be defined by $\phi(x, t) = \phi(f_\sigma(x))$ where $f_\sigma : (\mathbb{D}^n, \mathbb{S}^{n-1}) \to (X^n, X^{n-1})$ is the characteristic map of σ. Then we can extend Φ to a map $\Phi' : \partial(\mathbb{D}^n \times I) \to Y^{G_\sigma}$ which represents $d(G_\sigma)(\sigma) \in \pi_n(Y^{G_\sigma})$ as usual(See Proposition 7.11 in [7] for more detail). Then the map $\theta : X^n \cup A \to Y$ defined by

$$\theta|_{X^{n-1}\cup A} = \phi|_{X^{n-1}\cup A} \text{ and}$$
$$\theta(x) = \Phi'(f_\sigma^{-1}(x), 1) \text{ for any } x \in \sigma$$

satisfies that $d = d_{\phi,\mathrm{Id},\theta}$. $\square$

Theorem 3.2.1 *The restriction of a G-map $\phi : X^n \cup A \to Y$ to $X^{n-1} \cup A$ can be extended to $X^{n+1} \cup A$ if and only if the cohomology class represented by the obstruction cocyle c_ϕ is zero.*

Proof: If $[c_\phi] = 0$, then there is a cochain $d \in C^n(X, A; \underline{\pi})$ such that $c_\phi = \delta d$. Then by above proposition, there is a G-map $\theta : X^n \cup A \to Y$ such that $d = d_{\theta,\mathrm{Id},\phi}$. Since $c_\phi = \delta d$, we have $c_\theta = 0$ by Lemma 3.2.1. Therefore we can extend θ to $X^{n+1} \cup A$ equivariantly.

On the other hand, if one can extend $\phi|_{X^{n-1}\cup A}$ to $\bar{\phi} : X^{n+1} \cup A \to Y$ equivariantly then $c_{\bar{\phi}|_{X^n\cap A}}$ is zero. Then $\delta d_{\phi,\mathrm{Id},\bar{\phi}|_{X^n\cap A}} = c_\phi - c_{\bar{\phi}|_{X^n\cap A}} = c_\phi$ and hence $[c_\phi] = 0$. $\square$

3.3 Obstruction theory for constructing G-fibrations

In this chapter, we introduce an obstruction theory for constructing G-fibrations over G-CW-complexes and we prove Theorem 1.0.1.

3.3.1 The obstruction cocycle

Let $\mathcal{H}$ be a family of subgroups of G which is closed under conjugation and taking subgroups. A family $\mathcal{F} = \{F_H\}_{H\in\mathcal{H}}$ of H-spaces is said to be a *compatible family* if for every $H, K \in \mathcal{H}$ with $H^a \leq K$ for some $a \in G$, F_H is H-homotopy equivalent to F_K. Here F_K is considered as an H-space with H-action given by $h \cdot y = a^{-1}hay$. Note that a fiber $p^{-1}(b)$ of the G-fibration $p : E \to B$ is a K-space for every $b \in B^K$. Moreover, when $H^a \leq K$, there is an H action on $p^{-1}(b)$ given by conjugation and the spaces $p^{-1}(b)$ and $p^{-1}(ab)$ are H-homotopy equivalent. Therefore, the set of fibers of G-fibration p has a natural compatible family structure.

Definition 3.3.1 *A compatible family $\mathcal{F} = \{F_H\}_{H\in\mathcal{H}}$ where $\mathcal{H}$ contains the isotropy subgroups of B is called the fiber of the G-fibration $p : E \to B$ if $F_H \simeq_H p^{-1}(b)$ for every $b \in B^H$. We say p has a fiber $\mathcal{F}$ of homotopy type of a (finite) G-CW-complex if F_H has the H-homotopy of a (finite) H-CW-complex for every $H \in \mathcal{H}$.*

Let $\mathcal{F}in(G)$ be the category whose objects are G-fibrations $p : E \to B$ over a G-CW-complex B with the total space E of the G-homotopy type of a G-CW-complex and with fiber $\mathcal{F} = \{F_H\}_{H\in\mathcal{H}}$ of the homotopy type of a finite G-CW-complex and whose morphisms are fiber preserving G-maps. The following proposition is a slight generalization of Proposition 2.3.1 and the same proof applies here.

Proposition 3.3.1 *Let $p : E \to B$ be a G-fibration over a G-CW-complex B with fiber $\mathcal{F}$ of the homotopy type of a finite G-CW-complex. Then the total space E has the G-homotopy type of a G-CW-complex, that is, $p \in \mathcal{F}in(G)$. Moreover, if B is finite dimensional, then E is G-homotopy equivalent to a finite dimensional G-CW-complex.*

From now on, let $\mathcal{H}$ be the family of subgroups of G which contains the isotropy subgroups of G-action on B and which is closed under conjugation and taking subgroups. Let $\mathcal{F} = \{F_H\}_{H\in\mathcal{H}}$ be a compatible family and $n \geq 1$. Since $\mathcal{F}$ is compatible, we can consider the universal K-fibration

$$u_K : E\mathrm{Aut}_K F_K \to B\mathrm{Aut}_K(F_K)$$

with trivial K-action on the base as an H-fibration with the fiber F_H via conjugation action whenever $H^a \leq K$. Let $\phi : B\mathrm{Aut}_K(F_K) \to B\mathrm{Aut}_H(F_H)$ be the classifying map of this fibration. Define a contravariant functor

$$\underline{\pi}_{n,\mathcal{F}} : \mathcal{O}r_{\mathcal{I}} \to \mathcal{A}bel$$

by letting

$$\underline{\pi}_{n,\mathcal{F}}(G/H) = \pi_n(B\mathrm{Aut}_H(F_H))$$

$$\underline{\pi}_{n,\mathcal{F}}(\hat{a}) = \pi_*(\phi) : \pi_n(B\mathrm{Aut}_K(F_K)) \to \pi_n(B\mathrm{Aut}_H(F_H))$$

where $\hat{a} : G/H \to G/K$ is given by $\hat{a}(H) = aK$.

Let $p = \{p : E \to B_n\}$ be a G-fibration over the n-skeleton of B with fiber $\mathcal{F} = \{F_H\}_{H\in\mathcal{H}}$. Then for every $H \in \mathcal{H}$, the map $p_H = \{p^{-1}(B_n^H) \to B_n^H\}$ is an H-fibration with fiber F_H. Let $\phi_{p,H} : B_n^H \to B\mathrm{Aut}_H F_H$ be the classifying map of p_H. As in the standard theory, let us define

$$\alpha_p \in \prod_{H\in\mathcal{H}} \mathrm{Hom}_{\mathbb{Z}}(C_{n+1}(B^H), \pi_n(B\mathrm{Aut}_H F_H))$$

by $\alpha_p(H)(\sigma) = [\phi_{p,H} \circ f_\sigma]$ for every $H \in \mathcal{H}$ and for every $(n+1)$-cell σ of B^H with an attaching map $f_\sigma : \mathbb{S}^n \to B_n^H$.

Recall that for α_p to be a cochain in the Bredon cohomology of B with local coefficients $\underline{\pi}_{n,\mathcal{F}}$ the following diagram must commute up to homotopy

$$\begin{array}{ccccc}
\mathbb{S}^n & \xrightarrow{f_\sigma} & B_n^K & \xrightarrow{\phi_{p,K}} & B\mathrm{Aut}_K F_K \\
\| & & {\scriptstyle\bar{a}}\downarrow & & {\scriptstyle\tilde{a}}\downarrow \\
\mathbb{S}^n & \xrightarrow{f_{a\sigma}} & B_n^H & \xrightarrow{\phi_{p,H}} & B\mathrm{Aut}_H F_H
\end{array}$$

for every $H, K \in \mathcal{H}$ with $H^a \leq K$ for some $a \in G$. The first square in the above diagram commutes since the attaching map $f_{a\sigma}$ of $a\sigma$ is given by $f_{a\sigma}(x) = af_\sigma(x)$.

For the second diagram let $u_H : E\mathrm{Aut}_H F_H \to B\mathrm{Aut}_H F_H$ be the universal H-fibration with trivial H-action on the base. Then as H-fibrations

$$\begin{aligned}(\widetilde{a} \circ \phi_{p,K})_* u_H &= \widetilde{a}_* \underbrace{(\phi_{p,K})_* u_H}_{U_K} \simeq \widetilde{a}_* u_K \simeq p_K \quad \text{and} \\ (\phi_{p,H} \circ \bar{a})_* u_H &= \bar{a}_* \underbrace{(\phi_{p,H})_* u_H}_{p_H} \simeq \bar{a}_*(p_H) \simeq p_K\end{aligned}$$

and hence $(\widetilde{a} \circ \phi_{K,p})_* u_H \simeq (\phi_{H,p} \circ \bar{a})_* u_H$. By Theorem 2.3.1 for H-fibrations over B_n^H with fiber F_H, maps $\widetilde{a} \circ \phi_{p,K}$ and $\phi_{p,H} \circ \bar{a}$ are homotopic. Therefore α_p is a cochain in $C^{n+1}(B, \underline{\pi}_{n,\mathcal{F}})$.

Proposition 3.3.2 *α_p is a cocyle.*

Proof: Note that $\alpha_p(H) \in C^{n+1}(B^H, \pi_n(B\mathrm{Aut}_H F_H))$ is the classical obstruction cocycle for extending the map $\phi_{p,H} : B_n^H \to B\mathrm{Aut}_H F_H$ to the $(n+1)$-th skeleton of B^H. Then by classical obstruction theory, we have

$$(\delta\alpha_p)(H)(\sigma) = \delta(\alpha(H))(\sigma) = 0$$

for any $(n+1)$-cell σ of B^H and hence $\delta\alpha_p = 0$. $\square$

From now on, we call α_p the *obstruction cocycle*.

3.3.2 The main theorem

In this section we prove Theorem 1.0.1.

Proposition 3.3.3 *An equivariant G-fibration $p \in \mathcal{F}in(G)$ extends to a G-fibration in $\mathcal{F}in(G)$ over B_{n+1} if and only if $\alpha_p = 0$.*

Proof: If the obstruction cocyle is zero, then $[\phi_{G_\sigma} \circ f_\sigma] = 0$ for any $(n+1)$-cell σ. By Theorem 2.3.1, the G_σ-fibration $p|_{\partial\sigma} : p^{-1}(\partial\sigma) \to \partial\sigma$ is G_σ-fiber homotopy equivalent to the trivial G_σ fibration $\varepsilon : F_{G_\sigma} \times \partial\sigma \to \partial\sigma$. Then by Corollary 2.5.2, the G_σ-fibration $p|_{\partial\sigma}$ extends to a G_σ-fibration $Cp|_{\partial\sigma} : C(p^{-1}(\partial\sigma)) \to \sigma$ over σ. Let us define

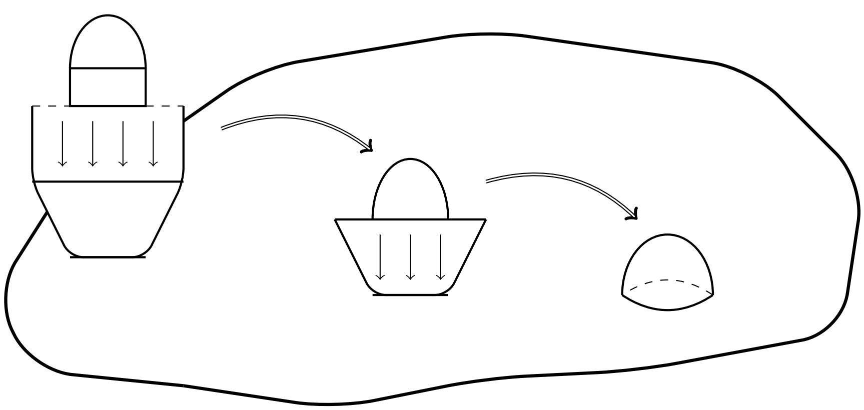
$$E' = (\coprod_{\sigma \in I_{n+1}} G \times_{G_\sigma} ((C(p^{-1}(\partial\sigma)) \cup_{i_1} p^{-1}(\partial\sigma) \times I)) \cup_{i_2} E$$

where I_{n+1} is the set of representative of G-orbits of $(n+1)$-cells and i_j are the corresponding inclusions for $j = 1, 2$. Let $q : E' \to B_{n+1}$ be defined by relations

$$\begin{aligned} q|_{C(p^{-1}(\partial\sigma))} &= Cp|_{\partial\sigma},\ q|_{Z_\sigma} = q_\sigma \\ q|_{p^{-1}(\partial\sigma)\times I} &= p|_\sigma \times \mathrm{id}_I \text{ and } q|E = p. \end{aligned}$$

Then by Theorem 2.1.2, the G-map q is a G-fibration.

Note that q has the fiber $\mathcal{F}$ and hence by Theorem 2.5.2, E' has the G-homotopy type of a G-CW-complex. Indeed, we can deform $(Cp)^{-1}(\partial\sigma) \cup_{i_1} p^{-1}(\partial\sigma) \times I \cup_{i_2} E$ to $\sigma \times F_{G_\sigma} \cup_{\beta_\sigma(f_\sigma \times \mathrm{id})} E$ relative to E via G_σ-map for each orbit representative $\sigma \in I_{n+1}$ as shown in the picture.



□

Proposition 3.3.3 proves the first part of Theorem 1.0.1. The second part of the theorem says that if α_p is cohomologous to zero, that is, $\alpha_p = \delta d$ for some cochain $d \in C^n(B, \underline{\pi}_{n,\mathcal{F}})$ then the G-fibration $p|_{B_{n-1}} : p^{-1}(B_{n-1}) \to B_{n-1}$ extends to a G-fibration over B_{n+1}. In order to prove this, we redefine p over the n-skeleton relative to the $(n-1)$-skeleton in such a way that the obstruction cocyle of the obtained G-fibration vanishes. Let us first fix some notation.

For a G-fibration $p : E \to B_n$ in $\mathcal{F}in(G)$, the map $p^H_{n-1} : p^{-1}(B^H_{n-1}) \to B^H_{n-1}$ is an H-fibration classified by $i_H \circ \phi_{p,H}$ where $i_H : B^H_{n-1} \to B^H_n$ is the inclusion. We write $\phi^{n-1}_{p,H} = i_H \circ \phi_{p,H}$ for short. Given two G-fibrations p_1 and p_2 over B_n with $p_1|_{B_{n-1}} \simeq p_2|_{B_{n-1}}$, the H-fibration $p^H_{1,n-1}$ is H-fiber homotopy equivalent to $p^H_{2,n-1}$ for every $H \le G$. Let $\Phi_H : B^H_{n-1} \times I \to B\mathrm{Aut}_H F_H$ be the homotopy between ϕ^{n-1}_{p,H_1} and ϕ^{n-1}_{p,H_2}. Then the composition $\Phi_H \circ \bar{a} : B^K_{n-1} \times I \to B\mathrm{Aut}_H F_H$ is a homotopy between $\phi^{n-1}_{p,H_1} \circ \bar{a}$ and $\phi^{n-1}_{p,H_2} \circ \bar{a}$. Similarly $\widetilde{a} \circ \phi^{n-1}_{K,p_1}$ and $\widetilde{a} \circ \phi^{n-1}_{K,p_2}$ are homotopic via the map $\widetilde{a} \circ \Phi_K : B^K_{n-1} \times I \to B\mathrm{Aut}_K F_K$. Since $\Phi_H \circ \bar{a}(-,0) = \phi^{n-1}_{p,H_1} \circ \bar{a}$ and $\widetilde{a} \circ \Phi_K(-,0) = \widetilde{a} \circ \phi^{n-1}_{K,p_1}$ are also homotopic, there is a homotopy between $\Phi_H \circ \bar{a}$ and $\widetilde{a} \circ \Phi_K$ by the following observation.

Lemma 3.3.1 *Let $H_i : X \times I \to Y$ be a homotopy between f_i and g_i for $i = 1, 2$. If f_1 and f_2 are homotopic then so are H_1 and H_2.*

Proof: Let $G : X \times I \to Y$ be the homotopy between f_1 and f_2. Then $H : X \times I \times I \to Y$ defined by

$$H(x,t,s) = \begin{cases} H_1(x,(1-4s)t), & 0 \le s < \frac{1}{4}; \\ G(x,4s-1) & \frac{1}{4} \le s < \frac{1}{2} \\ H_2(x,(2s-1)t) & \frac{1}{2} \le s \le 1. \end{cases}$$

is the desired homotopy between H_1 and H_2. $\square$

As a result, the following diagram commutes up to homotopy

$$\begin{array}{ccccc} \mathbb{S}^n & \xrightarrow{f_{a\tau \times I}} & B^K_{n-1} \times I & \xrightarrow{\Phi_K} & B\mathrm{Aut}_K F_K \\ \| & & \bar{a}\downarrow & & \widetilde{a}\downarrow \\ \mathbb{S}^n & \xrightarrow{f_{\tau \times I}} & B^H_{n-1} \times I & \xrightarrow{\Phi_K} & B\mathrm{Aut}_H F_H \end{array} \tag{3.2}$$

for every pair (H, K) with $H^a \le K$ and n-cell τ of B^H. Let us define the *deformation cochain* $d_{p_1,\Phi,p_2} \in \mathrm{Hom}(C_n(B), \underline{\pi}_{n,\mathcal{F}})$ by

$$d_{p_1,\Phi,p_2}(H)(\tau) = (-1)^{n+1}[\Psi_H \circ f_{\tau \times I}]$$

where $\Psi_H : (B^H \times I)_n = B^H_n \times \{0,1\} \cup B^H_{n-1} \times I \to B\mathrm{Aut}_H F_H$ is given by

$$\Psi_H(x,0) = \phi_{p,H_1}, \Psi_H(x,1) = \phi_{p,H_2}, \text{ and } \Psi_H(y,t) = \Phi_H(y,t)$$

for every $x \in B^H_n$, $y \in B^H_{n-1}$ and $t \in I$. This defines a cochain in $\mathrm{Hom}(C_n(B); \underline{\pi}_{n,\mathcal{F}})$ because of the commutativity of the diagram 3.2.

Proposition 3.3.4 $\delta d_{p_1,\Phi,p_2} = \alpha_{p_1} - \alpha_{p_2}$.

Proof: Let $o_\Psi \in C^{n+1}(B \times I, \underline{\pi}_{n,\mathcal{F}})$ be the obstruction cocycle to the extension of Ψ_H to $(B^H \times I)_{n+1}$. Then $\delta o_\Psi = 0$. On the other hand $o_\Psi(H)(e) = [\Psi_H \circ f_e]$ for every $(n+1)$-cell e of $B^H \times I$ and hence for any $(n+1)$-cell σ of B^H we have

$$\begin{aligned}
0 &= (\delta o_\Psi)(H)(\sigma \times I) \\
&= o_\Psi(H)(\partial\sigma \times I) + (-1)^{n+1}(o_\Psi(H)(\sigma \times \{1\}) - o_\Psi(H)(\sigma \times \{0\})) \\
&= [\Psi_H \circ f_{\partial\sigma \times I}] + (-1)^{n+1}([\phi_H \circ f_{\sigma \times \{1\}}] - [\phi_H \circ f_{\sigma \times \{0\}}]) \\
&= (-1)^{n+1}(d_{p_1,\Phi,p_2}(H)(\partial\tau) + [\phi_{p,H_2} \circ f_{\sigma \times \{1\}}] - [\phi_{p,H_1} \circ f_{\sigma \times \{0\}}]) \\
&= \delta d_{p_1,\Phi,p_2}(H)(\sigma) + \alpha_{p_2}(H)(\sigma) - \alpha_{p_1}(H)(\sigma)
\end{aligned}$$

which means that for every $H \in G$ and for every $(n+1)-$cell σ of B^H, we have

$$\delta d_{p_1,\Phi,p_2}(H)(\sigma) = (\alpha_{p_1} - \alpha_{p_2})(H)(\sigma)$$

and hence $\delta d_{p_1,\Phi,p_2} = \alpha_{p_1} - \alpha_{p_2}$. $\square$

The above proposition immediately implies the necessary part of the second part of Theorem 1.0.1. More precisely, if the G-fibration $p|_{B_{n-1}}$ extends to a G-fibration $q : E' \to B_{n+1}$ in $\mathcal{F}in(G)$ then $\delta d_{p,\mathcal{I},q|_{B_n}} = \alpha_p - \underbrace{\alpha_{q|_{B_n}}}_{0} = \alpha_p$ hence α_p represents a vanishing cohomology class. For the sufficiency part, we need the following observation.

Proposition 3.3.5 *Let $p : E \to B$ be a G-fibration in $\mathcal{F}in(G)$ over B_n and let $d \in \mathrm{Hom}(C_n(B), \underline{\pi}_{n,\mathcal{F}})$. Then there is a G-fibration $q \in \mathcal{F}in(G)$ over B_n such that $q|_{B_{n-1}} = p|_{B_{n-1}}$ and $d_{q,\mathrm{I},p} = d$ where $I_H : B^H_{n-1} \times I \to B\mathrm{Aut}_H F_H$ is given by $I_H(x,t) = \phi_{p,H}(x)$ for every $(x,t) \in B^H_{n-1} \times I$.*

Proof: Let $H \in G$ and τ be an n-cell of B^H. Let

$$\Psi_{\tau,H} : \mathbb{D}^n \times \{0\} \cup \mathbb{S}^{n-1} \times I \to B\mathrm{Aut}_H F_H$$

be given by

$$\Psi_{\tau,H}(x,t) = \Phi_{p,H}(f_\tau(x))$$

where $f_\tau : (\mathbb{D}^n, \mathbb{S}^{n-1}) \to (B_n^H, B_{n-1}^H)$ is the characteristic map. It is standard (see Lemma 7.11 [7]) to extend $\Psi_{\tau,H}$ to a map $\Psi'_{\tau,H} : \partial(\mathbb{D}^n \times I) \to B\mathrm{Aut}_H F_H$ representing $(-1)^{n+1} d(H)(\tau)$.

Let $p_\tau : E_\tau \to \mathbb{D}^n$ be the pullback of the universal G_τ-fibration u_{G_τ} via the map $\Psi'_{\tau,G_\tau}(-,1)$. Then

$$\begin{aligned} p_\tau|_{\mathbb{S}^{n-1}} &= (\Psi_{\tau,G_\tau}(-,1) \circ i)_* u_{G_\tau} = (\phi_{G_\tau} \circ f_\tau)_* u_{G_\tau} \\ &= (f_\tau)_*((\phi_{G_\tau})_* u_{G_\tau}) \simeq_{G_\tau} (f_\tau)_*(p_{G_\tau}). \end{aligned}$$

Therefore $p_\tau|_{\mathbb{S}^{n-1}}$ and $(f_\tau)_*(p_{G_\tau})$ are strongly G_τ-fiber homotopy equivalent. Let $q_\tau : Z_\tau \to \mathbb{S}^{n-1} \times I$ be the G-connection with $q_\tau|_{\mathbb{S}^{n-1}\times\{0\}} = p_\tau|_{\mathbb{S}^{n-1}}$ and $q_\tau|_{\mathbb{S}^{n-1}\times\{1\}} = (f_\tau)_*(p_{G_\tau})$. Then by Theorem 2.1.2 and by Corollary 2.5.2, the induced G-map

$$\begin{array}{c} E' = (\coprod_{\tau \in I_n} G \times_{G_\tau} (E_\tau \cup_{i_1} Z_\tau \times I \cup_{i_2} f_\tau^*(p^{-1}(\partial\tau)) \times I)) \cup_{i_3} E \\ \Big\downarrow{\scriptstyle q} \\ B_n \end{array}$$

is a G-fibration where i_j's are the corresponding inclusions. Clearly, we have $q|_{B_{n-1}} = p|_{B_{n-1}}$ and $d_{q,\mathrm{I},p} = d$. □

Now we can prove the other part of the main theorem as follows.

Proof of Theorem 1.0.1: It only remains to show that if α_p cohomologus to zero then there is a G-fibration over B_{n+1} which extends $p|_{B_{n-1}}$. Let $\alpha_p = \delta d$ for some $d \in \mathrm{Hom}(C_n(B), \underline{\pi}_n)$. Then there is a G-fibration q such that $d = d_{p,\mathcal{I},q}$ by Proposition 3.3.5. Since $\alpha_p = \delta d = \alpha_p - \alpha_q$, we have $\alpha_q = 0$ and hence q extends to a G-fibration over B_{n+1}. □

Chapter 4

The Euler class of the reduced regular representation

The aim of this section is to calculate the Euler classes of spherical fibrations associated to the reduced regular representations of finite groups. One motivation for these calculations is the following question raised by Reiner and Webb [18].

Problem 4.0.1 *Let $\Delta(G)$ be the simplicial complex whose simplices are nonempty subsets of G. Then the oriented chain complex of $\Delta(G)$ gives a $\mathbb{Z}G$-module extension of $\mathbb{Z}$ by $\widetilde{\mathbb{Z}}$ where $\widetilde{\mathbb{Z}}$ is a copy of integers on which G acts via the sign representation of the regular representation. Find all finite groups for which the extension class $\zeta_G \in \mathrm{Ext}^{|G|-1}_{\mathbb{Z}G}(\mathbb{Z}, \widetilde{\mathbb{Z}})$ of this extension is nonzero?*

The link between this problem and our calculations is an observation by Mandell which says that the Ext class of the complex $\Delta(G)$ is equal to the Euler class of the spherical fibration of the reduced regular representation of G.

It is shown in [18] that the extension class ζ_G is zero when G is not a p-group. Therefore we consider only p-groups. The following observation narrows the range for the search of p-groups with nonzero Euler class to abelian groups.

Theorem 4.0.1 *If G is a finite non-abelian group, then ζ_G is zero.*

For the abelian groups, our calculations lead us the following result:

Theorem 4.0.2 *Let G be a finite abelian group. Then, ζ_G is nonzero if and only if G is either an elementary abelian p-group or is isomorphic to $\mathbb{Z}/9$, $\mathbb{Z}/4 \times \mathbb{Z}/4$, or $(\mathbb{Z}/2)^n \times \mathbb{Z}/4$ for some integer $n \geq 0$.*

Theorem 4.0.1 and Theorem 4.0.2 together completes the proof of Theorem 1.0.2 stated in the introduction.

4.1 Euler class of a real representation

Let V be a real representation of G. Choose an arbitrary G-inner product on V and let $S(V)$ be the set of all unit vectors in V with respect to this inner product. Associated to the G-space $S(V)$, there is a spherical fibration $\xi_V : EG \times_G S(V) \to BG$ with fibers $S(V)$. Let $\mathbb{Z}(\xi_V)$ be the one dimensional integral representation of G given by

$$\mathrm{sgn}(V) : G \longrightarrow O(V) \xrightarrow{\det} \mathbb{Z}/2.$$

This one dimensional representation is called the *sign representation* of V. We say ξ_V is orientable if the sign representation of V is trivial. Otherwise we call it non-orientable. If V is a real representation with a fixed orientation, we denote the module $\mathbb{Z}(\xi_V)$ by $\widetilde{\mathbb{Z}}$ and define the *Euler class* $e(V) \in H^n(BG, \widetilde{\mathbb{Z}})$ as the first obstruction for the existence of a section for the sphere bundle $EG \times_G S(V) \to BG$ associated to V. Equivalently, the Euler class $e(V)$ is the first obstruction $o_n \in H^n_G(EG, \widetilde{\mathbb{Z}})$ for finding a G-map $f : EG \to S(V)$. We refer reader [10], for more details about the Euler classes of real representations.

Let Y be a finite G-CW-complex which has the homology of a sphere, say of dimension $n-1$. A *polarization* of Y is defined as a pair of isomorphisms $\varphi : H_0(Y) \to \mathbb{Z}$ and $\psi : H_{n-1}(Y) \to \widetilde{\mathbb{Z}}$. Associated to each polarized complex there is a unique k-invariant $\zeta \in H^n(BG, \widetilde{\mathbb{Z}})$ defined as follows: Note that given a polarized G-CW-complex Y with polarizations φ and ψ, we get an extension of $\mathbb{Z}G$-modules of the form

$$\varepsilon_Y : 0 \to \widetilde{\mathbb{Z}} \to C_{n-1}(Y) \to \cdots \to C_0(Y) \to \mathbb{Z} \to 0$$

using the polarizations at the ends of the extension to get $\mathbb{Z}$ and $\widetilde{\mathbb{Z}}$. This defines a unique extension class $\zeta(Y, \varphi, \psi) \in \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, \widetilde{\mathbb{Z}})$. The k-invariant is defined as the corresponding class in $H^n(BG, \widetilde{\mathbb{Z}})$.

We can fix the polarization $\varphi : H_0(Y) \to \mathbb{Z}$ by letting it to be the one given by augmentation map $C_0(Y) \to \mathbb{Z}$. For the unit sphere $S(V)$ of a real representation V, the polarization $\psi : H_{n-1}(S(V)) \to \mathbb{Z}$ corresponds to an orientation of V. Since V comes with a fixed orientation, there is a unique extension class $\zeta_V \in \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, \widetilde{\mathbb{Z}})$ associated to $S(V)$.

Proposition 4.1.1 *Let V be an n-dimensional real representation of G with a fixed orientation. Consider the exact sequence*

$$\varepsilon_V : \ 0 \to \widetilde{\mathbb{Z}} \to C_{n-1}(S(V)) \to \cdots \to C_0(S(V)) \to \mathbb{Z} \to 0$$

obtained by applying the polarizations coming from the fixed orientation of V. Then, the image of the extension class $\zeta_V \in \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, \widetilde{\mathbb{Z}})$ of this extension is equal to the Euler class $e(V)$ under the canonical isomorphism $\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, \widetilde{\mathbb{Z}}) \cong H^n(BG, \widetilde{\mathbb{Z}})$.

Proof: Since $\pi_i(S(V)) = 0$ for $i \leq n-1$, we can construct a G-map $f_{n-1} : EG^{(n-1)} \to S(V)$ in such a way that the induced map on 0-th homology is identity. Recall that the Euler class $e(V)$ is the obstruction to extending f_{n-1} to a G-map $f_n : EG^{(n)} \to S(V)$. By obstruction theory, this obstruction class is represented by a cocycle in $\mathrm{Hom}_G(C_n(EG), H_{n-1}(S(V))$ which is defined by the composition

$$o_n : C_n(EG) = \oplus_{\sigma_n} H_n(\sigma_n, \partial\sigma_n) \xrightarrow{\ \partial\ } \oplus_{\sigma_n} H_{n-1}(\partial\sigma_n) \xrightarrow{H_{n-1}(f)} H_{n-1}(S(V)).$$

Now, consider the following commutative diagram:

$$
\begin{tikzcd}
 & \oplus H_n(\sigma_n, \partial\sigma_n) \arrow[d, equal] & & \oplus H_{n-1}(\sigma_{n-1}, \partial\sigma_{n-1}) \arrow[d, equal] & & & & \\
 & C_n(EG) & & C_{n-1}(EG) & & & & \\
\cdots \arrow[r] & C_n(EG) \arrow[r] \arrow[dr] \arrow[ddddr, bend right, "o_n"'] & & C_{n-1}(EG) \arrow[r] \arrow[ddd, "f_{n-1}"] & \cdots \arrow[r] & C_0(EG) \arrow[r] \arrow[ddd, "f_0"] & \mathbb{Z} \arrow[r] \arrow[ddd, equal, "\mathrm{id}"] & 0 \\
 & & \oplus H_{n-1}(\partial\sigma_n) \arrow[ur] \arrow[dr] \arrow[ddd] & & & & & \\
 & 0 \arrow[ur] & & 0 & & & & \\
 & & & C_{n-1}(S(V)) \arrow[r] & \cdots \arrow[r] & C_0(S(V)) \arrow[r] & \mathbb{Z} \arrow[r] & 0 \\
 & & H_{n-1}(S(V)) \arrow[ur] & & & & & \\
 & 0 \arrow[ur] & & & & & &
\end{tikzcd}
$$

It is clear from this diagram that o_n is the lifting of the identity, so $e(V)$ corresponds to the extension class of the bottom extension under the isomorphism $H^n(G, H_{n-1}(S(V))) \cong \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, H_{n-1}(S(V)))$. Note that since V has a fixed orientation, there is a canonical isomorphism $H_{n-1}(S(V), \mathbb{Z}) \cong \widetilde{\mathbb{Z}}$ which we can use to replace $H_{n-1}(S(V), \mathbb{Z})$ with $\widetilde{\mathbb{Z}}$ in the above argument and hence $e(V) = \xi_V$. $\square$

Now define ζ_X as the extension class of the extension

$$\varepsilon_X : \qquad 0 \to \widetilde{\mathbb{Z}} \to C_{n-1}(\Delta(X)) \to \ldots \cdots \to C_0(\Delta(X)) \to \mathbb{Z} \to 0$$

where $\Delta(X)$ is the subset complex of X. Clearly this extension is equivalent to the extension

$$0 \to H_{n-1}(\partial\Delta(X)) \to C_{n-1}(\partial\Delta(X)) \to \cdots \to C_0(\partial\Delta(X)) \to \mathbb{Z} \to 0$$

where $\partial\Delta(X)$ denotes the boundary of the subset complex $\Delta(X)$.

Lemma 4.1.1 ([3], Lemma 2.2) *Let $S(I_X)$ denotes the unit sphere of the augmentation ideal $I_X = \ker\{\mathbb{R}X \to \mathbb{R}\}$ where X is a finite G-set and $|\partial\Delta(X)|$ denotes the realization of the boundary of the subset complex $\Delta(X)$. Then, there is a G-homeomorphism between the topological spaces $S(I_X)$ and $|\partial\Delta(X)|$.*

Proof: Let $X = \{x_0, \ldots, x_n\}$. We can regard I_X as the normal space of the vector $(1, \ldots, 1)$ and x_i as the i-th unit vector. Let v_i be the unit vector of the projection of x_i into I_X. Then the set $\{v_0, \ldots, v_n\}$ is an affinely independent set of vectors in $S(I_X)$. Let $\Delta'(X)$ be the n-simplex with vertex set $\{v_0, \ldots, v_n\}$. Let us define a G-homeomorphism $\phi : \Delta(X) \to \Delta'(X)$ by $\phi(x_i) = v_i$. Since ϕ sends $\partial\Delta(X)$ to the boundary of $\Delta'(X)$, it induces a G-homeomorphism between the associated topological spaces $S(I_X)$ and $|\partial\Delta(X)|$. $\square$

Theorem 4.1.1 *The Ext class ζ_X is equal to the Euler class $e(I_X)$ of the augmentation module I_X under the canonical isomorphism $\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, \widetilde{\mathbb{Z}}) \cong H^n(BG, \widetilde{\mathbb{Z}})$.*

Proof: Fix an ordering of elements in X so that we have a fixed orientation throughout. By Lemma 4.1.1, the chain complexes of $\partial\Delta(X)$ and $S(I_X)$ are chain homotopic. This means that the extension class ζ_X also represents the following exact sequence

$$0 \to \widetilde{\mathbb{Z}} \to C_{n-1}(S(I_X)) \to \cdots \to C_0(S(I_X)) \to \mathbb{Z} \to 0.$$

However by Proposition 4.1.1, this extension is represented by the Euler class $e(I_X)$. Therefore, $\zeta_X = e(I_X)$ as desired. $\square$

Although in our calculations we use the group cohomology with twisted coefficients, the Euler class itself often lies in the integral cohomology of G (with no twisting). In fact, Reiner and Webb showed that ζ_G lies in the cohomology with twisted coefficients if and only if G has a nontrivial cyclic Sylow 2-subgroup (see Lemma 5.4, [18]). In particular, if G is a p-group, then the Euler class is twisted only when G is a cyclic 2-group in which case the coefficients are given by the unique nontrivial map $G \to \mathbb{Z}/2$.

4.2 Proof of Theorem 4.0.1

The key ingredient in the proof of Theorem 4.0.1 is the following reduction argument.

Proposition 4.2.1 *If the Euler class ζ_G is nonzero, then the Euler class $\zeta_{H/K}$ is nonzero for every subquotient H/K of G.*

Proof: In [18], it is shown that for any subgroups $K \leq H \leq G$, the extension class $\zeta_{G/K}$ is equivalent to the cup product of $\zeta_{G/H}$ and $\mathcal{N}_H^G(\zeta_{H/K})$ where $\mathcal{N}$ denotes the Evens' norm map (see Proposition 7.13 in [18]). Therefore, for subgroups $1 \leq K \leq H \leq G$ we have

$$\zeta_G = \zeta_{G/H} \cdot \mathcal{N}_H^G(\zeta_{H/K} \cdot \mathcal{N}_K^H(\zeta_K)).$$

From this it follows immediately that if $\zeta_{H/K} = 0$ for some subquotient H/K of G, then $\zeta_G = 0$. $\square$

Proposition 4.2.1 implies that a minimal counterexample to Theorem 4.0.1 should have all proper subquotients abelian. First we classify such non-abelian p-groups and then we show that $\zeta_G = 0$ for all groups in the list. This means that there can not be any counterexamples to Theorem 4.0.1, hence it completes the proof. The classification of all nonabelian p-groups whose proper subquotients are all abelian is given as follows:

Proposition 4.2.2 *If G is a non-abelian p-group whose proper subquotients are all abelian then either G has order p^3 or G is isomorphic to the modular p-group M_{p^k} for some $k \geq 4$.*

Proof: Since every group of order less than p^3 is abelian and nonabelian groups of order p^3 obviously satisfy the assumption of the theorem, we can assume that G is a nonabelian p-group of order $|G| > p^3$ whose proper subquotients are all abelian. Let c be a central element of order p in G. Since $G/\langle c\rangle$ is abelian and G is nonabelian, $\langle c\rangle$ is the commutator group of G. Similarly, any central subgroup of order p is the commutator group and hence G has only one central subgroup of order p. This means that the center $Z(G)$ of G is cyclic. Note that G has an element of order p which is not central because otherwise G has a unique subgroup of order p which implies that G is either cyclic or a generalized quaternion group (see Theorem 4.3 in [5]). However these groups do not satisfy our assumption. So, G has an element of order p which is not central, say a. Let s be an element of G which does not commute with a. Since the subgroup generated by s and a is non-abelian, we must have $G = \langle a, s\rangle$. Note that $asa^{-1}s^{-1} = c^t$ for some

$t \not\equiv 0 \bmod p$. This gives $as^pa^{-1} = c^{tp}s^p = s^p$, so s^p is central in G. This forces the Frattini subgroup of G to be the subgroup generated by c and s^p. Thus, the Frattini subgroup is central, and hence cyclic. Since $|G| > p^3$, the element s^p cannot be trivial. So, we have $c^r = s^{p^{k-2}}$ for some $r \not\equiv 0 \bmod p$ where p^{k-1} is the order of s. Note that we can also assume $r = t = 1$ by replacing a and c with appropriate powers of themselves. Therefore, G has a presentation

$$G = \langle a, s \mid a^p = s^{p^{k-1}} = 1,\ asa^{-1} = s^{p^{k-2}+1} \rangle.$$

Hence, it is isomorphic to the modular group of order p^k with $k \geq 4$. $\Box$

Therefore in order to prove Theorem 4.0.1, it suffices to show that the Euler class ζ_G is zero for all the groups listed in Proposition 4.2.2. We will use different arguments for p is odd and $p = 2$. Let us first deal with the case where $p = 2$.

Lemma 4.2.1 *The Euler class ζ_G is zero when $G \cong Q_8$ or $G \cong M_{2^k}$ with $k \geq 3$.*

Proof: In both cases the Frattini subgroup $\Phi(G)$ of G is cyclic and central, and the quotient $G/\Phi(G)$ is isomorphic to the elementary abelian group of order 4. Therefore G has a central extension of the form

$$0 \to \Phi(G) \to G \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to 0.$$

Let us consider the Lyndon-Hochschild-Serre spectral sequence corresponding to this extension. By Proposition 7.2 in [33], the generator μ of the group $H^3(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z}) \cong \mathbb{Z}/2$ is in the image of the differential d_3. This means that $\inf^G_{G/\Phi(G)}(\mu)$ is zero in $H^3(G, \mathbb{Z})$.

The mod 2 reduction of μ is $x_1x_2(x_1 + x_2)$ which is the top Stiefel-Whitney class of $I_{G/\Phi(G)}$. Since the mod 2 reduction is injective for elementary abelian groups, this gives $\mu = e(I_{G/\Phi(G)})$. Therefore $\mathrm{Inf}^G_{G/\Phi(G)}\zeta_{G/\Phi(G)} = 0$, and hence $\zeta_G = 0$. $\Box$

Now, consider the case $p > 2$. The modular p-group M_{p^k} with $k \geq 4$ has an abelian subgroup isomorphic to $\mathbb{Z}/p^2 \times \mathbb{Z}/p$. Thus, the fact that $\zeta_G = 0$ for $G = M_{p^k}$ with $k \geq 4$ is a consequence of the following lemma.

Lemma 4.2.2 *The Euler class ζ_G is zero when $G \cong \mathbb{Z}/p^2 \times \mathbb{Z}/p$ and p is odd.*

Proof: Let $G = \langle a, b \mid a^{p^2} = b^p = 1,\ ab = ba \rangle$. Then $H^*(G,\mathbb{Z}) = \mathbb{Z}[\alpha,\beta] \otimes \wedge_p(\chi)$ where $\deg\alpha = \deg\beta = 2$, $\deg\chi = 3$ and $p^2\alpha = p\beta = p\chi = 0$ (see [15]). Since the Chern class c_1 defines an isomorphism $\mathrm{Hom}(G, \mathbb{C}^\times) \cong H^2(G,\mathbb{Z})$, we can consider the generators α and β of $H^2(G,\mathbb{Z})$ as the Chern classes of the representations $V_1 : a \longrightarrow \omega$, $b \longrightarrow 1$ and $V_2 : a \longrightarrow 1$, $b \longrightarrow \omega^p$ where ω is the primitive p^2-th root of unity. With this notation, the Chern class $c_1(V_3)$ of the one dimensional complex representation $V_3 : a \longrightarrow \omega^p$, $b \longrightarrow 1$ is equal to $p\alpha$.

Let W_2 and W_3 be the underlying 2-dimensional real representations of V_2 and V_3 respectively. Then we have

$$e(W_2 \oplus W_3) = e(W_2)e(W_3) = c_1(V_2)c_1(V_3) = p\alpha\beta = 0.$$

Since $W_2 \oplus W_3$ is a direct summand of the augmentation module I_G, this gives $\zeta_G = 0$. $\Box$

It remains to consider the nonabelian groups of order p^3 for $p > 2$. The following lemma solves the problem for this case and hence completes the proof of Theorem 4.0.1.

Lemma 4.2.3 *Let G be a nonabelian p-group of order p^3 with $p > 2$. Then, the Euler class ζ_G is zero.*

Proof: If G is a p-group of order p^3 with $p > 2$, then G is either isomorphic to the extra-special group E_{p^3} of exponent p or to the modular group M_{p^3} of exponent p^2. In both cases, the exponent of $H^i(G,\mathbb{Z})$ is p when i is not divisible by $2p$ (see [15]). Therefore, the mod p reduction map $H^{p^3-1}(G,\mathbb{Z}) \longrightarrow H^{p^3-1}(G,\mathbb{F}_p)$ is injective. So, it is sufficient to show that the mod p reduction of ζ_G is zero.

In the mod p cohomology of the groups E_{p^3} and M_{p^3}, there are relations of the form $xy = 0$ and $\beta(x) + xy = 0$ respectively, where x, y denote the generators of one dimensional cohomology for each group. If we apply the operator $\beta P \beta$ to these relations, then we obtain

$$\beta(x)\prod_{j=0}^{p-1}\beta(jx+y)=0$$

This product is a factor of the mod p reduction of ζ_G. Hence, $\zeta_G = 0$. □

4.3 Calculations for some abelian 2-groups

In this section we calculate the Euler class ζ_G for some small abelian 2-groups.

Proposition 4.3.1 *The Euler class ζ_G is zero when $G \cong \mathbb{Z}/8$.*

Proof: Let H be the maximal subgroup of G. We have $\zeta_G = \zeta_{G/H} \cdot e(W)$ where W is the direct sum of all irreducible two dimensional representations of G. The Euler class $\zeta_{G/H}$ is represented by the extension

$$0 \to \widetilde{\mathbb{Z}} \to \mathbb{Z}[G/H] \to \mathbb{Z} \to 0.$$

Consider the long exact sequence associated with this short exact sequence:

$$\begin{tikzcd}
\cdots \arrow[r] & H^6(G,\widetilde{\mathbb{Z}}) \arrow[r] & H^6(H,\mathbb{Z}) \arrow[r, "\mathrm{tr}"] \arrow[d, equal] & H^6(G,\mathbb{Z}) \arrow[r, "\cdot\zeta_{G/H}"] \arrow[d, equal] & H^7(G,\widetilde{\mathbb{Z}}) \arrow[r] & \cdots \\
& & \mathbb{Z}/|H|\mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z}/|G|\mathbb{Z} & &
\end{tikzcd}$$

It is clear from the above diagram that to show that $\zeta_G = 0$, it suffices to show that $e(W) \in H^6(G,\mathbb{Z})$ is divisible by 2. Note that $W = W_1 + W_2 + W_3$ where W_i is the two dimensional real representation such that the action is given by $2\pi i/8$ degree rotation. Also note that if $\alpha = e(W_1)$, then $e(W_j) = j\alpha$ for all $j = 1, 2, 3$. This gives $e(W) = 6\alpha^3$ which is divisible by 2 as desired. □

The calculation above shows that the Euler class ζ_G is zero for all cyclic 2-groups with order greater or equal to 8. An easy calculation shows that the Euler class for $\mathbb{Z}/4$ is not zero. We will show later that ζ_G is not zero also when

$G = \mathbb{Z}/4 \times \mathbb{Z}/4$. The next group we consider is $G = (\mathbb{Z}/4)^2 \times \mathbb{Z}/2$. We show that the Euler class for this group is zero. For this calculation, we need the structure of the cohomology of the group $\mathbb{Z}/4 \times \mathbb{Z}/4$ with integer coefficients. In [23], Townsley completely describes the integral cohomology of all abelian groups. We quote the result from [23], but since what we need is a very special case of Townsley's calculations, we provide a proof for the convenience of the reader.

Proposition 4.3.2 (Townsley [23]) *Let $G = \mathbb{Z}/4 \times \mathbb{Z}/4$. Then,*

$$H^*(G, \mathbb{Z}) \cong \mathbb{Z}/4[\mu_1, \mu_2, \mu_{12}]/\langle \mu_{12}^2 - 2\mu_1\mu_2(\mu_1 + \mu_2)\rangle.$$

Proof: Let us consider the Lyndon-Hochschield-Serre spectral sequence with E_2-page

$$E_2^{p,q} \cong H^p(G/K, H^q(K, \mathbb{Z})) \Rightarrow H^{p+q}(G, \mathbb{Z})$$

where K is a cyclic subgroup of G of order 4. Let t_1 and t_2 be the generators of $H^*(K, \mathbb{Z})$ and $H^*(G/K, \mathbb{Z})$. Since $H^2(G, \mathbb{Z}) \cong H^2(G/K, \mathbb{Z}) \oplus H^2(K, \mathbb{Z})$, we have $d_2(t_1) = 0$, which implies that $d_2 = 0$. By dimension reasons $d_i = 0$ for any $i \geq 2$, hence the spectral sequence collapses at E_2-page. Thus, the Poincaré series of $H^*(G, \mathbb{Z})$ is given by

$$P_{H^*(G,\mathbb{Z})}(t) = \frac{1 + t^3}{(1 - t^2)^2}.$$

Since $E_2^{1,2} = H^1(G/K, H^2(K, \mathbb{Z})) \cong \mathbb{Z}/4$, the cohomology ring $H^*(G, \mathbb{Z})$ has at least three generators. Let μ_1 and μ_2 be the generators of degree 2 and let μ_{12} be the generator of degree 3. Without loss of generality, we can assume $\mathrm{res}^G_K \mu_1 = t_1$ and $\mathrm{Inf}^G_{G/K} t_2 = \mu_2$. We claim that μ_1 and μ_2 are algebraically independent. Indeed, if

$$f(\mu_1, \mu_2) = \sum_{i=0}^{k} a_i \mu_1^i \mu_2^{k-i}$$

is a relation with the smallest degree then the restriction of $f(\mu_1, \mu_2)$ to the subgroup K gives $a_k t_1^k = 0$ and hence $a_k = 0$. Therefore, $f(\mu_1, \mu_2) = \mu_2 g(\mu_1, \mu_2)$ for some polynomial $g(\mu_1, \mu_2)$ with smaller degree. Since μ_2 is a nonzero divisor, this gives $g(\mu_1, \mu_2) = 0$ which contradicts the minimality of $f(\mu_1, \mu_2)$. Therefore, μ_1 and μ_2 are algebraically independent as claimed.

Let us assume for the moment that $\mu_{12}^2 = 2\mu_1\mu_2(\mu_1+\mu_2)$ is the only relation for the generators μ_1, μ_2 and μ_{12}. It implies that

$$S = \mathbb{Z}/4[\mu_1,\mu_2,\mu_{12}]/\langle \mu_{12}^2 - 2\mu_1\mu_2(\mu_1+\mu_2)\rangle$$

is a subring of $H^*(G,\mathbb{Z})$. On the other hand, S and $H^*(G,\mathbb{Z})$ have the same Poincaré series. So, we obtain $H^*(G,\mathbb{Z}) = S$ as desired. Now we prove that $\mu_{12}^2 = 2\mu_1\mu_2(\mu_1+\mu_2)$ is the only relation. Since μ_{12} has an odd degree, we have $2\mu_{12}^2 = 0$. This implies that $(\mu_{12})^2 = 2f(\mu_1,\mu_2)$ for some polynomial $f(\mu_1,\mu_2) \in H^6(G,\mathbb{Z})$. It is easy to show that μ_{12}^2 is not zero by considering the spectral sequence associated to the extension

$$0 \to \mathbb{Z}/2\times\mathbb{Z}/2 \to G \to \mathbb{Z}/2\times\mathbb{Z}/2 \to 0.$$

So, we can assume $f(\mu_1,\mu_2) = a_1\mu_1^3 + a_2\mu_2^3 + a_3\mu_1^2\mu_2 + a_4\mu_1\mu_2^2$ where at least one of the a_i's is nonzero. Since the restriction of μ_{12} to any cyclic subgroup H of G is zero, we get $a_1 = a_2 = 0$ and $a_3 = a_4 = 1$. So, $\mu_{12}^2 = 2\mu_1\mu_2(\mu_1+\mu_2)$. Suppose now that there is another relation. Then, it must be of the form

$$\mu_{12}\cdot g(\mu_1,\mu_2) + h(\mu_1,\mu_2) = 0,$$

but this is impossible since the degree of μ_1 and μ_2 are 2 and the degree of μ_{12} is 3. $\square$

Now we are ready to do the following calculation.

Proposition 4.3.3 *The Euler class ζ_G is zero when $G \cong (\mathbb{Z}/4)^2\times\mathbb{Z}/2$.*

Proof: The Euler class ζ_G includes $\inf_{G/\Phi(G)}^G \zeta_{G/\Phi(G)}$ as a factor. So, it is enough to prove that this factor is zero. The cohomology ring $H^*(G/\Phi(G),\mathbb{Z})$ is generated by the elements u_I where the indices I run through the subsets of $\{1,2,3\}$. The mod 2 reduction of u_I is given by the formula

$$m_2(u_I) = \Big(\prod_{i\in I} x_i\Big)\Big(\sum_{i\in I} x_i\Big)$$

where x_1, x_2 and x_3 are the generators of $H^*(G/\Phi(G),\mathbb{Z}/2)$. By direct calculation, one can show that the mod 2 reduction of $u_1^2u_{23} + u_2^2u_{13} + u_3^2u_{12}$ is equal to the product of one dimensional classes, hence it is equal to the top Stiefel-Whitney class of $I_{G/\Phi(G)}$. Since the reduction modulo 2 is an injective map for elementary abelian groups, we conclude that

$$\zeta_{G/\Phi(G)} = u_1^2u_{23} + u_2^2u_{13} + u_3^2u_{12}.$$

Now we show that the inflation of this element is zero. Let a, b, c be the generators of G with $a^4 = b^4 = c^2 = 1$. Suppose μ_1, μ_2 and μ_3 are the generators of $H^2(G,\mathbb{Z}) = \mathrm{Hom}(G,\mathbb{C}^\times)$ which are dual to a, b, c respectively. We can choose the generators u_1, u_2, u_3 for $H^2(G/\Phi(G),\mathbb{Z})$ in a compatible way and assume that $\mathrm{Inf}^G_{G/\Phi(G)}u_i = 2\mu_i$ for $i = 1, 2$ and $\mathrm{Inf}^G_{G/\Phi(G)}u_3 = \mu_3$. Since the exponent of $H^*(G,\mathbb{Z})$ is 4, we get $\mathrm{Inf}^G_{G/\Phi(G)}u_i^2 = 0$ for $i = 1, 2$. Hence

$$\mathrm{Inf}^G_{G/\Phi(G)}\big(u_1^2u_{23} + u_2^2u_{13} + u_3^2u_{12}\big) = \mu_3^2 \cdot \mathrm{Inf}^G_{G/\Phi(G)}u_{12}.$$

On the other hand, we have

$$\mathrm{Inf}^G_{G/\Phi(G)}u_{12} = \mathrm{Inf}^G_{\overline{G}}\mathrm{Inf}^{\overline{G}}_{\overline{G}/\Phi(\overline{G})}u_{12} = \mathrm{Inf}^G_{\overline{G}}\, 2\mu_{12} = 2\mathrm{Inf}^G_{\overline{G}}\, \mu_{12}$$

where $\overline{G} = G/\langle c\rangle \simeq \mathbb{Z}/4\times\mathbb{Z}/4$ and μ_{12} is the generator of $H^3(\overline{G},\mathbb{Z}) = \mathbb{Z}/4$. Since $2\mu_3 = 0$, we get $\mathrm{Inf}^G_{G/\Phi(G)}\big(u_1^2u_{23} + u_2^2u_{13} + u_3^2u_{12}\big) = 0$. This completes the proof. □

4.4 Proof of Theorem 4.0.2

We first consider abelian 2-groups. In the previous section we showed that if G is isomorphic to $\mathbb{Z}/8$ or $(\mathbb{Z}/4)^2 \times \mathbb{Z}/2$, then the Euler class of the augmentation module I_G is zero. This implies that ζ_G is zero if G has a subquotient isomorphic to $\mathbb{Z}/8$ or $(\mathbb{Z}/4)^2 \times \mathbb{Z}/2$. But, the only abelian 2-groups that do not have any such subquotients are either elementary abelian or isomorphic to $\mathbb{Z}/4 \times \mathbb{Z}/4$ or $(\mathbb{Z}/2)^n \times \mathbb{Z}/4$ for some n. This proves one direction of Theorem 4.0.2 for 2-groups. For the other direction, we need to show that ζ_G is nonzero for these groups. We start with the calculation of ζ_G for $G = \mathbb{Z}/4 \times \mathbb{Z}/4$.

Proposition 4.4.1 *Let $G = \mathbb{Z}/4 \times \mathbb{Z}/4$. Then, the Euler class ζ_G is nonzero.*

Proof: Let $G = \langle a, b\rangle$, and let μ_1, μ_2 be the generators of $H^2(G, \mathbb{Z})$ dual to a and b respectively, and let μ_{12} be a generator of $H^3(G, \mathbb{Z})$. We have $I_G \cong \mathrm{Inf}^G_{G/\Phi(G)} I_{G/\Phi(G)} \oplus W$ where W is the direct sum of all irreducible two dimensional real representations of G. Since $G/\Phi(G)$ is an elementary abelian group of order 4, it follows from an argument similar to that used in the proof of Proposition 4.3.3 that $\zeta_{G/\Phi(G)} = u_{12}$. Hence we get

$$e(\mathrm{Inf}^G_{G/\Phi(G)} I_{G/\Phi(G)}) = \mathrm{Inf}^G_{G/\Phi(G)} \zeta_{G/\Phi(G)} = \mathrm{Inf}^G_{G/\Phi(G)} u_{12} = 2\mu_{12}.$$

Now, we calculate the Euler class of W. Every two dimensional real representation of G is the underlying real representation of a one dimensional complex representation. If θ is the real representation associated to the complex representation $\rho : G \to \mathbb{C}^\times$, then $e(\theta) = c_1(\rho)$. Note that for each two dimensional real representation, the kernel is a cyclic group of order 4. In fact, there is a one-to-one correspondence between two dimensional real representations of G and its cyclic subgroups of order 4. The cyclic subgroups of G are $\langle a\rangle$, $\langle ab^2\rangle$, $\langle ab\rangle$, $\langle ab^3\rangle$, $\langle a^2b\rangle$, and $\langle b\rangle$. Therefore

$$e(W) = \mu_2(2\mu_1 + \mu_2)(\mu_1 + \mu_2)(\mu_1 + 3\mu_2)\mu_1(\mu_1 + 2\mu_2) = \mu_1^2\mu_2^2(\mu_1 + \mu_2)^2$$

and hence $\zeta_G = 2\mu_{12}\mu_1^2\mu_2^2(\mu_1 + \mu_2)^2$. It is clear from Proposition 4.3.2 that this class is not zero in $H^*(G, \mathbb{Z})$. $\square$

It remains to show that the Euler class is nonzero when G is either elementary abelian or isomorphic to $(\mathbb{Z}/2)^n \times \mathbb{Z}/4$ for some $n \geq 0$. For these groups we show that the Euler class is nonzero by showing that its mod 2 reduction is nonzero. Recall that the mod 2 reduction of the Euler class of a real representation V is equal to the top Stiefel-Whitney class $w_{\mathrm{top}}(V)$ of V. To conclude that $w_{\mathrm{top}}(I_G)$ is nonzero, we give an explicit formula for it in terms of the generators of the cohomology ring of G. It is often more convenient to express the formula for $w_{\mathrm{top}}(I_G)$ in terms of the polynomial f where

$$f(a_1, a_2, \ldots, a_m) = \prod_{(\alpha_1,\ldots,\alpha_m)\in(\mathbb{F}_2)^m\setminus\{0\}} (\alpha_1 a_1 + \cdots + \alpha_m a_m)$$

for tuples $(a_1, \ldots, a_m)$. The formula for the top Stiefel-Whitney class of an elementary abelian 2-group appears in many places (see, for example, Turygin [26]). In this case, we have

$$w_{\mathrm{top}}(I_G) = f(x_1, \ldots, x_n)$$

where $\{x_1, \ldots, x_n\}$ is a set of generators of the cohomology ring $H^*(BG, \mathbb{F}_2)$. Note that this is the top Dickson invariant of the polynomial algebra $\mathbb{F}_2[x_1, \ldots, x_n]$. In particular, the Euler class ζ_G is nonzero when G is an elementary abelian 2-group.

Now, we perform a similar calculation for the group $G = (\mathbb{Z}/2)^n \times \mathbb{Z}/4$. We should note that it is possible to conclude that the top Stiefel-Whitney class is nonzero for these groups without obtaining an explicit formula, but we believe that the formula itself might also be useful. Before stating the result, we need some further notation. Let V_i be the one dimensional nontrivial representation inflated from the i-th term in the product of $\mathbb{Z}/2$'s and let W_{n+1} be the irreducible two dimensional representation inflated from the $\mathbb{Z}/4$ term of the product. Notice that the cohomology of the group G with $\mathbb{F}_2$ coefficients is

$$H^*(G, \mathbb{F}_2) \cong \mathbb{F}_2[x_1, \ldots, x_n, s] \otimes \wedge[t]$$

where $w_{\mathrm{top}}(V_i) = x_i$ for $1 \leq i \leq n$, $w_{\mathrm{top}}(V_{n+1}) = t$, and $w_{\mathrm{top}}(W_{n+1}) = s$. With this notation the formula for $w_{\mathrm{top}}(I_G)$ can be expressed as follows:

Proposition 4.4.2 *Let $G \cong (\mathbb{Z}/2)^n \times \mathbb{Z}/4$ with $n \geq 0$. Then,*

$$w_{\mathrm{top}}(I_G) = f(x_1, \ldots, x_n, t)\frac{f(x_1^2, \ldots, x_n^2, s)}{f(x_1^2, \ldots, x_n^2)}.$$

In particular, $w_{\mathrm{top}}(V)$ is nonzero in $H^*(G, \mathbb{F}_2)$.

Proof: Let V be the direct sum of all the nontrivial one dimensional real representations of G. Since each nontrivial one dimensional real representation is a tensor product of elements contained in some nonempty subset of $\{V_1, \cdots, V_{n+1}\}$, we have $w_{\mathrm{top}}(V) = f(x_1, \cdots, x_n, t)$. On the other hand, $I_G = V \oplus W$ where

$$W = \bigoplus_{(\gamma_1, \cdots, \gamma_n) \in S_n} V_1^{\gamma_1} \otimes \cdots \otimes V_n^{\gamma_n} \otimes W_{n+1}.$$

Here S_n is the set of n-tuples $(\gamma_1, \cdots, \gamma_n)$ such that $\gamma_i \in \{1,2\}$ for all i. By the tensor product formula for one dimensional real vector bundles, we have

$$w(V_1^{\gamma_1} \otimes \cdots \otimes V_n^{\gamma_n}) = 1 + \alpha_1 x_1 + \cdots + \alpha_n x_n$$

where α_i denotes the mod 2 reduction of γ_i for each i. Using the splitting principle, we can regard the r-th Stiefel-Whitney class of the vector bundle W_{n+1} as the r-th elementary symmetric function of indeterminates a_1 and a_2 so that $w_1(W_{n+1}) = a_1 + a_2 = 0$ and $w_2(W_{n+1}) = a_1 a_2 = s$. Then, we have

$$\begin{aligned} w(V_1^{\gamma_1} \otimes \cdots \otimes V_n^{\gamma_n} \otimes W_{n+1}) &= \prod_{i=1}^{2} (1 + \alpha_1 x_1 + \cdots + \alpha_n x_n + a_i) \\ &= 1 + \alpha_1 x_1^2 + \cdots + \alpha_n x_n^2 + s. \end{aligned}$$

Thus, the top Stiefel-Whitney class of W is given by

$$\begin{aligned} w_{\text{top}}(W) &= \prod_{(\gamma_1, \cdots, \gamma_n) \in S_n} w_{\text{top}}(V_1^{\gamma_1} \otimes \cdots \otimes V_n^{\gamma_n} \otimes W_{n+1}) \\ &= \prod_{(\alpha_1, \cdots, \alpha_n) \in (\mathbb{F}_2)^n} (\alpha_1 x_1^2 + \cdots + \alpha_n x_n^2 + s) = \frac{f(x_1^2, \dots, x_n^2, s)}{f(x_1^2, \dots, x_n^2)}. \end{aligned}$$

The formula for $w_{\text{top}}(I_G)$ follows from the identity $w_{\text{top}}(I_G) = w_{\text{top}}(V) w_{\text{top}}(W)$. Note that since $t^2 = 0$, we can rewrite the top Stiefel-Whitney class as

$$w_{\text{top}}(I_G) = t\big(f(x_1, \dots, x_n)\big)^2 \frac{f(x_1^2, \dots, x_n^2, s)}{f(x_1^2, \dots, x_n^2)}.$$

From this it is clear that $w_{\text{top}}(I_G)$ is nonzero in $H^*(G, \mathbb{F}_2)$. $\square$

This completes the proof of Theorem 4.0.2 for 2-groups. Now, we consider the case $p > 2$. We begin with the calculations for cyclic groups. Let $G = \langle g \rangle$ be a cyclic group of order p^n with $p > 2$. All nontrivial representations of G are two dimensional which are the underlying real representations of one dimensional complex representations. A complete list of corresponding complex representations can be given as $V_j : g \to \omega^j$ where ω is the p^n-th root of unity and $1 \leq j \leq \frac{p^n - 1}{2}$. We can take $\alpha = c_1(V_1)$ as the generator of $H^2(G, \mathbb{Z}) \cong \mathbb{Z}/p^n$, then we have $c_1(V_j) = j\alpha$ for all j. This gives

$$e(I_G) = \prod_{j=1}^{\frac{p^n-1}{2}} c_1(V_j) = (\frac{p^n - 1}{2})! \ \alpha^{\frac{p^n-1}{2}}.$$

From this we conclude the following:

Lemma 4.4.1 *Let p be an odd prime and G be a cyclic p-group. Then the Euler class ζ_G is nonzero if and only if G has order p or is isomorphic to $\mathbb{Z}/9$.*

Proof: Suppose that G has order p^n. Then, $H^{2k}(G, \mathbb{Z}) \simeq \mathbb{Z}/p^n$ for all $k \geq 1$. It follows that $e(I_G) = 0$ if and only if

$$(\frac{p^n - 1}{2})! \equiv 0 \pmod{p^n}.$$

It is easy to see that this equation holds for all p and n except when $n = 1$ or when $p = 3$ and $n = 2$. $\square$

The above lemma implies that the Euler class ζ_G of an abelian p-group with $p > 3$ vanishes if G is not elementary abelian. For $p = 3$, we need to be more careful. Since ζ_G is not zero for $\mathbb{Z}/9$, we need to consider the next possibility, which is $\mathbb{Z}/9 \times \mathbb{Z}/3$. But, this is the special case of Lemma 4.2.2, so $\zeta_G = 0$ in this case as well. This proves one direction of Theorem 4.0.2 for $p > 2$. For the other direction, we need to show that ζ_G is nonzero when G is an elementary abelian p-group with $p > 2$. This follows easily from the structure of cohomology of elementary abelian p-groups since the 2-dimensional classes in $H^*((\mathbb{Z}/p)^n, \mathbb{Z})$ generate a polynomial subalgebra. So, the proof of Theorem 4.0.2 is complete.

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